UNITED STATES
NAVAL POSTGRADUATE SCHOOL

MUTUAL IMPEDANCE OF RHOMBIC ANTENNAS SPACED IN TANDEM

---BY---

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PROFESSOR OF ELECTRONICS

A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
BUSHIPS PROJECT ORDER NO. 10731/52

DECEMBER 1, 1952

TECHNICAL REPORT NO. 7

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REPORT NO. 4

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ABSTRACT

Upon examining the formulas for self and mutual impedances of antennas, it is found that while the mutual impedance formula for separately driven collinear standing wave antennas may be used directly in determining the radiation impedance when such antennas are connected in cascade, certain modifications must be made in the case of travelling wave antennas under similar circumstances. Accordingly, formulas are derived for rhombic antennas spaced in tandem and are modified to permit the determining of the radiation impedance of two identical rhombic antennas connected in cascade.

IN TANDEM VS. IN CASCADE

The mesh equations for two coupled antennas are

$$\begin{aligned} Z_{11}I_1 + Z_{12}I_2 &= V_1 \\ Z_{21}I_1 + Z_{22}I_2 &= V_2 \\ Z_{12} &= Z_{21} \end{aligned} \tag{1}$$

The radiation impedance parameters are given by certain double line integrals[1], in which both paths are along the same wire for a self impedance, whereas one path is along each wire for a mutual impedance.

In the case of an open wire full wave antenna fed at a current antinode, a conventional method for determining the driving point impedance is to set

$$V_2 = 0, \quad I_2 = -I_1 \tag{2}$$

and add the two equations, giving,

$$Z_{in} = 2(Z_{11} - Z_{12}) \tag{3}$$

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1. J. G. Chaney, ''A critical study of the circuit concept'', J. Appl. Phys., 22, 12, 1429 (1951).

The validity of the above procedure rests in the fact that not only are the integration paths for equation (3) exactly the same as they would be if the paths were directly set up for the self impedance of the full wave antenna, but the assumed currents satisfy the restriction

$$I_1^* I_2 = Re(I_1^* I_2) = -|I_1|^2 \tag{4}$$

On the other hand, if the reference currents were not in time phase and equation (4) were not satisfied, say

$$I_2 = I_1 \, exp(jkh) , \tag{5}$$

the division by $|I_1|^2$, as required in the formulation of a self impedance from the expression for the complex power, instead of by $I_1^* I_2$ as required for the formulation of the mutual impedance, would require the current factor[1] $Re[f_1(P_1)^* f_2(P_2)]$, to be replaced by the factor,

$$Re[f_1(P_1)^* f_2(P_2) exp(jkh)] \; .$$

In other words, a mutual impedance formula for two individually driven travelling wave antennas would require modification before it would be permissible to write the radiation impedance of two such identical antennas driven in cascade by the formula

$$Z_r = 2(Z_{11} + Z_{12}) \tag{6}$$

RHOMBIC ANTENNAS IN TANDEM

It has been shown[2] that the mutual impedance of two terminated rhombic antennas is given by

$$jkZ_{12} = \int_1 \int_2 e(r_{12}) \left[\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \cos \theta(s_1, s_2) \right] g(ks_1, ks_2) ds_1 ds_2 \tag{7}$$

in which

$g(ks_1, ks_2) = Re[f_1(ks_1)^* f_2(ks_2)]$

$e(r_{12}) = r_{12}^{-1} exp(-jkr_{12})$

$f_1(ks_1)$ = *current distribution function along antenna one*

$f_2(ks_2)$ = *current distribution function along antenna two*

$\theta(s_1, s_2)$ = *angle between directions along the two antennas*

r_{12} = *distance between points on the two antennas*

$k = 2\pi/\lambda$

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2. J. G. Chaney, "Simplification for mutual impedance of certain antennas", U. S. Naval Postgrad. School Tech. Rpt. nd 6, Nov., 1952.

For two identical, coaxial, and coplanar rhombic antennas, the same corresponding sides are parallel as in the case of the paths which were used for finding the self radiation impedance of a rhombic antenna[3]. Since the current distribution functions are the same as those for the two paths along the same antenna, it follows that the integrations in equation (7) need only to be carried out over the nonparallel wires[3].

Let each leg of the rhombus be of length l, and let the vertex angles at the generator be 2α. Then, after postulating unattenuated travelling waves of current along each rhombic antenna, equation (7) becomes

$$Z_{12} = j120\sin^2\alpha\Big[\int_0^l\int_0^l \cos k(x_1-x_2)e(r_{12})dx_1dx_2+\int_0^l\int_0^l \cos k(x_1-x_3)e(r_{13})dx_1dx_3 - \int_0^l\int_0^l \cos k(x_1+x_4)e(r_{14})dx_1dx_4-\int_0^l\int_0^l \cos k(x_1+x_5)e(r_{15})dx_1dx_5 \tag{8}$$

or changing to the exponential form,

$$Z_{12} = -j60\sin^2\alpha[I_1+I_2+I_3+I_4-2I_5-2I_7] \tag{9}$$

in which

$$I_1 = \int_0^l\int_0^l \exp[-jk(x_5+x_1+r_{15})]r_{15}^{-1}dx_5dx_1 \tag{10}$$

$$I_2 = \int_0^l\int_0^l \exp[\,jk(x_5+x_1-r_{15})]r_{15}^{-1}dx_5dx_1 \tag{11}$$

$$I_3 = \int_0^l\int_0^l \exp[-jk(x_4+x_1+r_{14})]r_{14}^{-1}dx_4dx_1 \tag{12}$$

$$I_4 = \int_0^l\int_0^l \exp[\,jk(x_4+x_1-r_{14})]r_{14}^{-1}dx_4dx_1 \tag{13}$$

$$I_5 = \int_0^l\int_0^l \exp[\,jk(x_3-x_1-r_{13})]r_{13}^{-1}dx_3dx_1 \tag{14}$$

$$I_7 = \int_0^l\int_0^l \exp[\,jk(x_2-x_1-r_{12})]r_{12}^{-1}dx_2dx_1 \tag{15}$$

in which x_0 is the distance between the driving points, and in which

$$r_{12} = [x_1^2+x_2^2+x_0^2-2x_0(x_1-x_2)\cos\alpha-2x_1x_2\cos2\alpha]^{\frac{1}{2}} \tag{16}$$

$$r_{13} = [x_1^2+x_3^2+x_0^2+2x_0(x_1-x_3)\cos\alpha \qquad -2x_1x_3\cos2\alpha]^{\frac{1}{2}} \tag{17}$$

$$r_{14} = [x_1^2+x_5^2+x_0^2+2x_0{}^2(x_1+x_4)\cos\alpha+2x_1x_4\cos2\alpha]^{\frac{1}{2}} \tag{18}$$

$$r_{15} = [x_1^2+x_5^2+x_0^2-2x_0(x_1+x_5)\cos\alpha+2x_1x_5\cos2\alpha]^{\frac{1}{2}} \tag{19}$$

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3. J. G. Chaney, ''Free space radiation impedance of a rhombic antenna'', U. S. Naval Postgrad. School Tech. Rpt. no. 4, May, 1952.

Let

$$d_1 = [x_0^2 + (2l\sin a)^2]^{\frac{1}{2}}$$

$$d_2 = [x_0^2 + l^2 - 2lx_0 c\,csa]^{\frac{1}{2}}$$

$$d_3 = [x_0^2 + l^2 + 2lx_0\cos a]^{\frac{1}{2}}$$

and

$$Z_{12} = R_{12} + jX_{12}\ .$$

After evaluating the definite integrals in equation (9), the following formula is obtained,

$$\begin{aligned}
&R_{12}/60 = \cos k(x_0\sec a)\{4Cik(x_0\sec a+l+d_3)+4Cik(x_0\sec a+l-d_3)+4Cik(x_0\sec a-l+d_2)\\
&+4Cik(x_0\sec a-l-d_2)-4Cik[(\sec a+1)x_0]-4Cik[x_0(\sec a-1]-2Cik(x_0\sec a+d_1)\\
&-2Cik(x_0\sec a-d_1)-Cik[x_0(\sec a-1)+2l(1-\cos a)]-Cik[x_0(\sec a-1)-2l(1-\cos a)]\\
&-Cik[x_0(\sec a+1)+2l(1+\cos a)]-Cik[x_0(\sec a+1)-2l(1+\cos a)]\}\\
&+\sin k(x_0\sec a)\{4Sik(x_0\sec a+l+d_3)+4Sik(x_0\sec a+l-d_3)+4Sik(x_0\sec a-l+d_2)\\
&+4Sik(x_0\sec a-l-d_2)-4Sik[x_0(\sec a-1)]-4Sik[x_0(\sec a+1)]-2Sik(x_0\sec a+d_1)\\
&-2Sik(x_0\sec a-d_1)-Sik[x_0(\sec a+1)+2l(1+\cos a)]-Sik[x_0(\sec a+1)-2l(1+\cos a)]\\
&-Sik[x_0(\sec a-1)+2l(1-\cos a)]-Sik[x_0(\sec a-1)-2l(1-\cos a)]\}\\
&-\cos k(x_0\cos a)\{2Cik(d_3+l+x_0\cos a)+2Cik(d_3-l-x_0\cos a)+2Cik(d_2+l-x_0\cos a)\\
&+2Cik(d_2-l+x_0\cos a)-4Cik[x_0(1+\cos a)]-4Cik[x_0(1-\cos a)]\} \qquad (20)\\
&+\sin k(x_0\cos a)\{2Sik(d_3+l+x_0\cos a)+2Sik(d_3-l-x_0\cos a)+2Sik(d_2+l-x_0\cos a)\\
&+2Sik(d_2-l+x_0\cos a)-4Sik[x_0(1+\cos a)]-4Sik[x_0(1-\cos a)]\}\\
&+\cos k(x_0\cos a+2l\sin^2 a)\{Cik(d_1+x_0\cos a+2l\sin^2 a)+Cik(d_1-x_0\cos a-2l\sin^2 a)\\
&-2Cik(d_2+x_0\cos a-l\cos 2a)-2Cik(d_2-x_0\cos a+l\cos 2a)+Cik[(x_0-2l\cos a)(1+\cos a)]\\
&+Cik[(x_0-2l\cos a)(1-\cos a)]\}\\
&+\sin k(x_0\cos a+2l\sin^2 a)\{Sik(d_1+x_0\cos a+2l\sin^2 a)-Sik(d_1-x_0\cos a-2l\sin^2 a)\\
&-2Sik(d_2+x_0\cos a-l\cos 2a)+2Sik(d_2-x_0\cos a+l\cos 2a)+Sik[(x_0\cos a-l\cos 2a)(1+\cos a)]\\
&--Sik[(x_0-2l\cos a l k1-\cos a)]\}\\
&+\cos k(x_0\cos a-2l\sin^2 a)\{Cik(d_1+x_0\cos a-2l\sin^2 a)+Cik(d_1-x_0\cos a+2l\sin^2 a)\\
&-2Cik(d_3+x_0\cos a+l\cos 2a)-2Cik(d_3-x_0\cos a-l\cos 2a)+Cik[(x_0+2l\cos a)(1+\cos a)]\\
&+Cik[(x_0+2l\cos a)(1-\cos a)]\}\\
&+\sin k(x_0\cos a-2l\sin^2 a)\{Sik(d_1+x_0\cos a-2l\sin^2 a)-2Sik(d_3+x_0\cos a+l\cos 2a)\\
&-Sik(d_1-x_0\cos a+2l\sin^2 a)+2Sik(d_3-x_0\cos a-l\cos 2a)+Sik[(x_0+2l\cos a)(1+\cos a)]\\
&-Sik[(x_0+2l\cos a)(1-\cos a)]\}
\end{aligned}$$

$$
\begin{aligned}
X_{12} = {} & \cos(x_0\sec\alpha)\{4Sik(x_0\sec\alpha+l-d_3)-4Sik(x_0\sec\alpha+l+d_3)-4Sik(x_0\sec\alpha+l-d_2) \\
& +4Sik(x_0\sec\alpha-l-d_2)+4Sik[x_0(\sec\alpha+1)]-4Sik[x_0(\sec\alpha-1)]+2Sik(x_0\sec\alpha+d_1) \\
& -2Sik(x_0\sec\alpha-d_1)+Sik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]+Sik[x_0(\sec\alpha+1)+2l(1+\cos\alpha)] \\
& -Sik[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]-Sik[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\} \\
& -\sin k(x_0\sec\alpha)\{4Cik(x_0\sec\alpha+l-d_3)-4Cik(x_0\sec\alpha+l+d_3)-4Cik(x_0\sec\alpha-l+d_2) \\
& +4Cik(x_0\sec\alpha-l-d_2)+4Cik[x_0(\sec\alpha-1)]-4Cik[x_0(\sec\alpha-1)]+2Cik(x_0\sec\alpha+d_1) \\
& -2Cik(x_0\sec\alpha-d_1)+Cik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]+Cik[x_0(\sec\alpha+1)+2l(1+\cos\alpha)] \\
& -Cik[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]-Cik[x_0\sec\alpha-1)+2l(1-\cos\alpha)]\} \\
& +\cos(x_0\cos\alpha)\{2Sik(d_3-l-x_0\cos\alpha)+2Sik(d_3+l+x_0\cos\alpha)+2Sik(d_2-l+x_0\cos\alpha) \\
& +2Sik(d_2+l-x_0\cos\alpha)-4Sik[x_0(1+\cos\alpha)]-4Sik[x_0(1-\cos\alpha)]\} \\
& +\sin k(x_0\cos\alpha)\{2Cik(d_3-l-x_0\cos\alpha)-2Cik(d_3+l+x_0\cos\alpha)-2Cik(d_1-l+x_0\cos\alpha) \\
& +2Cik(d_1+l-x_0\cos\alpha)+4Cik[x_0(1+\cos\alpha)]-4Cik[x_0(1-\cos\alpha)]\} \\
& -\cos k(x_0\cos\alpha+2l\sin^2\alpha)\{Sik(d_1+x_0\cos\alpha+2l\sin^2\alpha)+Sik(d_1-x_0\cos\alpha-2l\sin^2\alpha) \\
& -2Sik(d_2+x_0\cos\alpha-l\cos2\alpha)-2Sik(d_2-x_0\cos\alpha+l\cos2\alpha)+Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)] \\
& +Sik[(x_0-2l\cos\alpha)(1-\cos\alpha)]\} \\
& +\sin k(x_0\cos\alpha+2l\sin^2\alpha)\{Cik(d_1+x_0\cos\alpha+2l\sin^2\alpha)-Cik(d_1-x_0\cos\alpha-2l\sin^2\alpha) \\
& -2Cik(d_2+x_0\cos\alpha-l\cos2\alpha)+2Cik(d_2-x_0\cos\alpha+l\cos2\alpha)+Cik[(x_0-2l\cos\alpha)(1+\cos\alpha)] \\
& -Cik[(x_0-2l\cos\alpha)(1-\cos\alpha)]\} \\
& -\cos k(x_0\cos\alpha-2l\sin^2\alpha)\{Sik(d_1+x_0\cos\alpha-2l\sin^2\alpha+Sik(d_1-x_0\cos\alpha+2l\sin^2\alpha) \\
& -2Sik(d_3+x_0\cos\alpha+l\cos2\alpha)-2Sik(d_3-x_0\cos\alpha-l\cos2\alpha)+Sik[(x_0+2l\cos\alpha)(1+\cos\alpha)] \\
& +Sik[(x_0+2l\cos)(1-\cos\alpha)]\} \\
& +\sin k(x_0\cos\alpha-2l\sin^2\alpha)\{Cik(d_1+x_0\cos\alpha-2l\sin^2\alpha)-Cik(d_1-x_0\cos\alpha+2l\sin^2\alpha) \\
& -2Cik(d_3+x_0\cos\alpha+l\cos2\alpha)+2Cik(d_3-x_0\cos\alpha-l\cos2\alpha)+Cik[(x_0+2l\cos\alpha)(1+\cos\alpha)] \\
& -Cik[(x_0+2l\cos\alpha)(1-\cos\alpha)]\}
\end{aligned} \tag{21}
$$

For two closely spaced identical rhombic antennas, equations (20) and (21) may be considerably simplified. At the frequency for which a rhombic antenna is given an optimum design, the leg length is almost always taken as an integral multiple of a half wave length with the parameter $2kl$ becoming an even multiple of π. Under these conditions, the formulas may be further simplified. However, when operating at a frequency different from the design frequency, $2kl$ is usually not an integral multiple of π, and the formulas must be more carefully examined in case it is desired to connect the antennas in cascade. Accordingly, select

$$x_0 = 2l\cos\alpha + b, \quad b<<2l\cos\alpha, \quad b<<l \tag{22}$$

let

$$d_4 = l(1+8\cos^2\alpha)^{\frac{1}{2}}$$

and simplify.

$$
\begin{aligned}
R_{12}/60 = {} & \cos 2kl\{C+\ln(kl\sin^2\alpha)+2\,Ci\,2kl-Ci\,4kl-4\,Ci\,[2kl(1+\cos\alpha)]-4\,Cik[2kl(1-\cos\alpha)] \\
& -Ci\,[4kl(1+\cos\alpha)]-Ci\,[4kl(1-\cos\alpha)]+4\,Cik(3l+d_4)+4\,Cik(3l-d_4)\} \\
& +\sin 2kl\{2\,Si\,2kl-Si\,4kl-4\,Si\,[2kl(1+\cos\alpha)]-4\,Si\,[2kl(1-\cos\alpha)]-Si\,[4kl(1+\cos\alpha)] \\
& -Si\,[4kl(1-\cos\alpha)]+4\,Sik(3l+d_4)+4\,Sik(3l-d_4)\} \\
& -2\cos(2kl\cos^2\alpha)\{Ci(2kl\sin^2\alpha)+Ci(2kl\cos^2\alpha)-2\,Ci\,[2kl\cos\alpha(1+\cos\alpha)] \\
& -2\,Ci\,[2kl\cos\alpha(1-\cos\alpha)]+Cik(d_4+2l\cos^2\alpha)+Cik(d_4-2l\cos^2\alpha)\} \\
& +2\sin(2kl\cos^2\alpha)\{Si(2kl\sin^2\alpha)+Si(2kl\cos^2\alpha)-2\,Si\,[(2kl\cos\alpha)(1+\cos\alpha)] \\
& -2\,Si\,[2kl\cos\alpha(1-\cos\alpha)]+Sik(d_4+2l\cos^2\alpha)+Sik(d_4-2l\cos^2\alpha)\} \\
& +\cos(2kl\cos 2\alpha)\{Ci\,(4kl\cos^2\alpha)+Ci(4kl\sin^2\alpha)+Cik[4kl\cos\alpha(1+\cos\alpha)] \\
& +Cik[4kl\cos\alpha(1-\cos\alpha)]-2\,Cik(d_4+4l\cos^2\alpha-l)-2\,Cik(d_4-4l\cos^2\alpha+l)\} \\
& +\sin(2kl\cos 2\alpha)\{Si\,(4kl\cos^2\alpha)-Si\,(4kl\sin^2\alpha)+Sik[4l\cos\alpha(1+\cos\alpha)] \\
& -Sik[4l\cos\alpha(1-\cos\alpha)]-2\,Sik(d_4+4l\cos^2\alpha-l)+2\,Sik(d_4-4l\cos^2\alpha+l)\}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
X_{12} = {} & \cos 2kl\{-2\,Si\,2kl+Si\,4kl+4\,Si\,[2kl(1+\cos\alpha)]-4\,Si\,[2kl(1-\cos\alpha)] \\
& +Si\,[4kl(1+\cos\alpha)]-Si\,[4kl(1-\cos\alpha)]-4\,Sik(3l+d_4)+4\,Sik(3l-d_4)\} \\
& -\sin 2kl\{C+\ln(kl\tan^2\alpha)-2\,Ci\,2kl+Ci\,4kl+4\,Ci\,[2kl(1+\cos\alpha)]-4\,Cik[2kl(1-\cos\alpha)] \\
& +Ci\,[4kl(1+\cos\alpha)]-Ci\,[4kl(1-\cos\alpha)]-4\,Cik(3l+d_4)+4\,Cik(3l-d_4)\} \\
& +2\cos(2kl\cos^2\alpha)\{Sik(2l\sin^2\alpha)+Si\,(2kl\cos^2\alpha)-2\,Si\,[2kl\cos\alpha(1+\cos\alpha)]-2\,Si\,[2kl\cos\alpha \times \\
& (1-\cos\alpha)]+Sik(d_4+2l\cos^2\alpha)+Sik(d_4-2l\cos^2\alpha)\} \\
& +2\sin(2kl\cos^2\alpha)\{Ci\,(2kl\sin^2\alpha)-Ci(2kl\cos^2\alpha)+2\,Ci\,[2kl\cos\alpha(1+\cos\alpha)] \\
& -2\,Ci\,[2kl\cos\alpha(1-\cos\alpha)]-Cik(d_4+2l\cos^2\alpha)+Cik(d_4-2l\cos^2\alpha)]\} \\
& -\cos(2kl\cos 2\alpha)\{Si\,(4kl\sin^2\alpha)+Si(4kl\cos^2\alpha)+Si\,[4kl\cos\alpha(1+\cos\alpha)] \\
& +Si\,[4kl\cos\alpha(1-\cos\alpha)]-2\,Sik(d_4+4l\cos^2\alpha-l)-2\,Sik(d_4-4l\cos^2\alpha+l)\} \\
& +\sin(2kl\cos 2\alpha)\{-Ci\,(4kl\sin^2\alpha)+Ci\,(4kl\cos^2\alpha)+Ci\,[4kl\cos\alpha(1+\cos\alpha)] \\
& -Ci\,[4kl\cos\alpha(1-\cos\alpha)]-2\,Cik(d_4+4l\cos^2\alpha-l)+2\,Cik(d_4-4l\cos^2\alpha+l)\}
\end{aligned}
\tag{24}
$$

in which $C = 0.5772\ldots$ is Euler's constant.

Upon selecting the leg length as an integral multiple of a half wave length, formulas (23) and (24) reduce to

$$
\begin{aligned}
R_{12}/60 = {} & C+ln(klsin^2\alpha)-Ci4kl+2Ci2kl-4Ci[2kl(1+cos\alpha)]-4Ci[2kl(1-cos\alpha)]\\
& -Ci[4kl(1+cos\alpha)]-Ci[4kl(1-cos\alpha)]+4Cik(3l+d_4)+4Ci(3l-d_4))\}\\
& -2cos(2klsin^2\alpha\{Ci(2klsin^2\alpha)+Ci(2klcos^2\alpha)-2Ci[2klcos\alpha(1+cos\alpha)]\\
& -2Ci[2klcox\alpha(1-cos\alpha)]+Cik(d_4+2lcox^2\alpha)+Cik(d_4-2lcos^2\alpha)]\}\\
& -2sin(2klsin^2\alpha)\{Sik(2klsin^2\alpha)+Si(2klcos^2\alpha)-2Si[2klcos\alpha(1+cos\alpha)]\\
& -2Si[2klcos\alpha(1-cos\alpha)]+Sik(d_4+2lcos^2\alpha)+Sik(d_4-2lcos^2\alpha)\}\\
& -cos(4klsin^2\alpha)\{-Ci(4klsin^2\alpha)-Ci(4klcos^2\alpha)-Ci[4klcos\alpha(1+cos\alpha)]\\
& -Ci[4klcos\alpha(1-cos\alpha)]+2Cik(d_4+4lcos^2\alpha-l)+2Cik(d_4-4lcos^2\alpha+l)\}\\
& -sin(4klsin^2\alpha)\{-Si(4klsin^2\alpha)+Si(4klcos^2\alpha)+Si[4klcos\alpha(1+cos\alpha)]\\
& -Si[4klcos\alpha(1-cos\alpha)]+2Sik(d_4-4lcos^2\alpha+l)-2Si(d_4+4lcos^2\alpha-l)\}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
X_{12}/60 = {} & Si4kl-2Si2kl+4Si[2kl(1+cos\alpha)]-4Si[2kl(1-cos\alpha)]+Si[4kl(1+cos\alpha)]\\
& -Si[4kl(1-cos\alpha)]+4Sik(3l-d_4)-4Sik(3l+d_4))\}\\
& +2cos(2klsin^2\alpha)\{Si(2klsin^2\alpha)+Si(2klcos^2\alpha)-2Si[2klcos\alpha(1+cos\alpha)]\\
& -2Si[2klcos\alpha(1-cos\alpha)]+Sik(d_4+2lcos^2\alpha)+Sik(d_4-2lcos^2\alpha)\}\\
& -2sin(2klsin^2\alpha)\{Ci(2klsin^2\alpha)-Ci(2klcos^2\alpha)+2Ci[2klcos\alpha(1+cos\alpha)]\\
& -2Ci]2klcos\alpha(1-cos\alpha)]-Cik(d_4+2lcos^2\alpha)+Ci(d_4-2lcos^2\alpha)\}\\
& +cos(4klsin^2\alpha)\{-Si(4klsin^2\alpha)-Si(4klcos^2\alpha)-Si[4klcos\alpha(1+cos\alpha)]\\
& -Si[4klcos\alpha(1-cos\alpha)]+2Sik(d_4+4lcos^2\alpha-l)+2Sik(d_4-4lcos^2\alpha+l)\}\\
& -sin(4klsin^2\alpha)\{-Ci(4klsin^2\alpha)+Ci(4klcos^2\alpha)+Ci[4klcos\alpha(1+cos\alpha)]\\
& -Ci[4klcos\alpha(1-cos\alpha)]+2Cik(d_4-4lcos^2\alpha+l)-2Cik(d_4+4lcos^2\alpha-l)\}
\end{aligned}
\tag{26}
$$

Formulas (20) and (21) are for any two rhombic antennas that are terminated, have equal leg lengths, are coplanar, and are coaxially spaced x_0 between driving points. Formulas (23) and (24) are for any two identical rhombic antennas which are closely spaced in tandem. Formulas (25) and (26) are for two identical rhombic antennas whose leg lengths are an integral multiple of a half wave length and which are closely spaced in tandem. It will be shown that formulas (25) and (26) may also be used for rhombic antennas connected in cascade even though the leg lengths are arbitrary.

RHOMBIC ANTENNAS IN CASCADE

If instead of solving for the phase of I_2 as in the case of separately driven antennas, the phase of I_2 is to be specified with respect to I_1 in accordance with equation (5), an examination of the integrations of equations (10) to (15), inclusive, reveals that formulas (20) and (21) require modification only in the sine and cosine coefficients of the sine integral and cosine integral functions within the braces. This modification consists

merely of adding the term kh to the argument of the sine and cosine coefficients. For example, $\cos k(x_0 \sec a)$ is replaced by $\cos k(h+x_0 \sec a)$, *etc.* The algebraic sign of h is always the same as that of x_0.

If two identical rhombic antennas are to be connected in cascade, in order that a form of Z_{12} suitable for substitution into equation (6) may be written, select $h=-2l$ and modify equations (23) and (24). This again gives the formulas (25) and (26). Thus the particular form given for the mutual impedance of two closely spaced rhombic antennas in equations (25) and (26) is also valid for antennas connected in cascade even though the leg lengths are not an even multiple of a half wave length.

Hence, it follows that Z_{12}, as given by the latter two equations, may be used with the previously derived formula for the self radiation impedance of a single rhombic antenna [3] for substitution into equation (6), giving the free space radiation impedance of two identical rhombic antennas in cascade.

Perhaps it should be emphasized that although the radiation impedance of a system of rhombic antennas determines the power radiated by the system, and is thereby useful in determining the gain of the system, it does not constitute the driving point impedance of the system. However, it does largely determine the attenuation of the current along the various legs of the system, and as a consequence does enter into the finding of the driving point impedance to a certain extent.

APPENDIX A

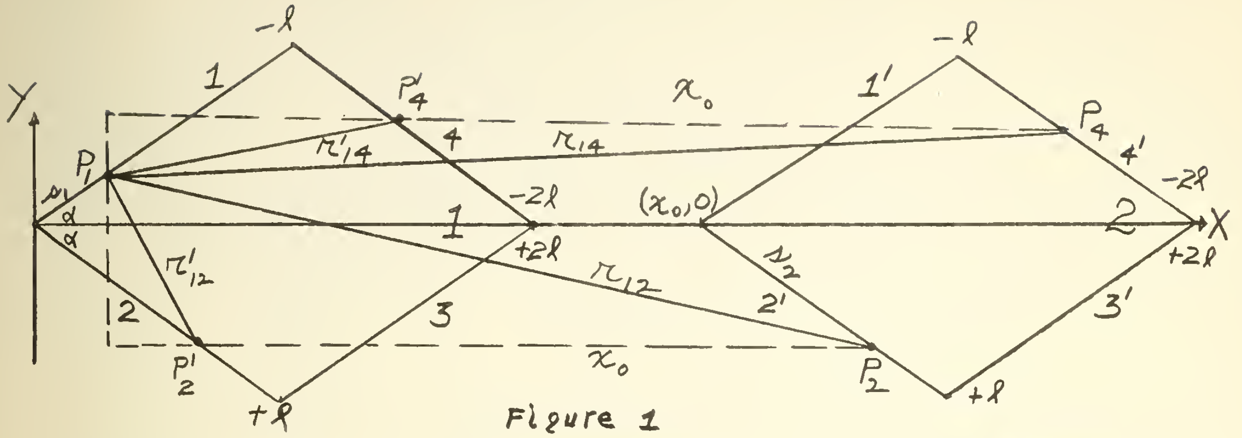

Figure 1

Variable s_1 is along rhombic 1, s_2 is along rhombic 2

$$-l \leq s_1 \leq 0 \qquad , \qquad 0 \leq s_2 \leq +l$$

$$-2l \leq s_2 \leq -l$$

Distance r_{12} is for paths $(1, 2')$

r_{14} is for paths $(1, 4')$

now, $$r_{12}' = \sqrt{s_1^2 + s_2^2 + 2 s_1 s_2 \cos 2\alpha}$$

Hence $$r_{12} = \sqrt{r_{12}'^2 + x_0^2 + 2 x_0 r_{12}' \frac{(s_2+s_1)\cos\alpha}{r_{12}'}}$$

$$= \sqrt{s_1^2 + s_2^2 + 2 x_0 (s_1 + s_2)\cos\alpha + 2 s_1 s_2 \cos 2\alpha + x_0^2}$$

or $$r_{12} = \sqrt{(s_1 + x_0 \cos\alpha + s_2 \cos 2\alpha)^2 + (x_0 \sin\alpha + s_2 \sin 2\alpha)^2}$$

Write $$r_{14}' = \sqrt{(s_1+l)^2 - 2(s_1+l)(s_2+l)\cos 2\alpha + (s_2+l)^2}$$

and $$r_{14} = \sqrt{(s_1+l)^2 + (s_2+l)^2 - 2(s_1+l)(s_2+l)\cos 2\alpha + x_0^2 - 2x_0(s_2-s_1)\mathrm{co}}$$

Hence $$r_{14} = \sqrt{[(s_1+l) + x_0\cos\alpha - (s_2+l)\cos 2\alpha]^2 + [x_0 \sin\alpha - (s_2+l)\sin 2\alpha]^2}$$

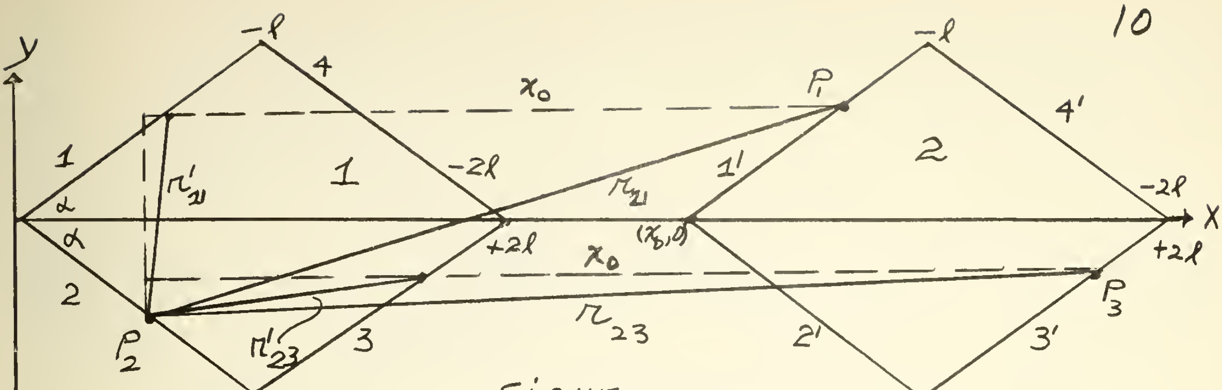

Figure 2

s_1 varies along rhombic 1, s_2 varies along rhombic 2

$$0 \le s_1 \le +\ell, \qquad -\ell \le s_2 \le 0$$
$$+\ell \le s_2 \le +2\ell$$

Distance r_{21} is for paths $(2, 1')$

r_{23} is for paths $(2, 3')$

$$r'_{21} = \sqrt{s_1^2 + s_2^2 + 2 s_1 s_2 \cos 2\alpha}$$

$$r_{21} = \sqrt{s_1^2 + s_2^2 + 2 s_1 s_2 \cos 2\alpha + x_0^2 - 2 x_0 (s_1 + s_2) \cos\alpha}$$

$$r_{21} = \sqrt{[s_1 - x_0 \cos\alpha + s_2 \cos 2\alpha]^2 + [x_0 \sin\alpha - s_2 \sin 2\alpha]^2}$$

$$r'_{23} = \sqrt{(\ell - s_1)^2 + (s_2 - \ell)^2 + 2(\ell - s_1)(s_2 - \ell)\cos 2\alpha}$$
$$= \sqrt{(s_1 - \ell)^2 + (s_2 - \ell)^2 - 2(s_1 - \ell)(s_2 - \ell)\cos 2\alpha}$$

$$r_{23} = \sqrt{(s_1 - \ell)^2 + (s_2 - \ell)^2 - 2(s_1 - \ell)(s_2 - \ell)\cos 2\alpha + 2 x_0 (s_2 - s_1)\cos\alpha + x_0^2}$$

$$r_{23} = \sqrt{[(s_1 - \ell) - x_0 \cos\alpha - (s_2 - \ell)\cos 2\alpha]^2 + [x_0 \sin\alpha + (s_2 - \ell)\sin 2\alpha]^2}$$

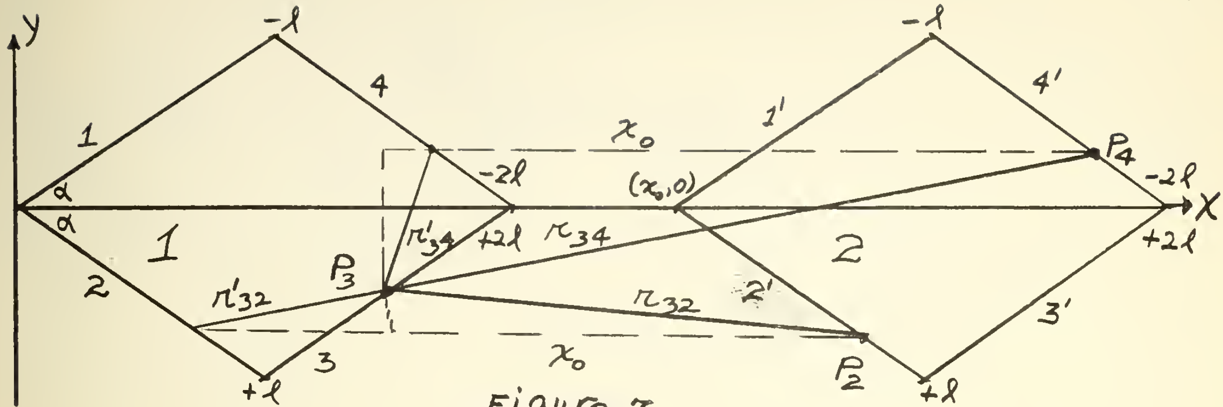

Figure 3

s_1 varies along rhombic 1, s_2 varies along rhombic 2

$+l \le s_1 \le 2l$

$0 \le s_2 \le +l$

$-2l \le s_2 \le -l$

Distance r_{32} is for paths (3, 2')

r_{34} is for paths (3, 4')

$$r'_{32} = \sqrt{(l-s_2)^2 + (s_1-l)^2 + 2(l-s_2)(s_1-l)\cos 2\alpha}$$

$$r_{32} = \sqrt{(s_1-l)^2 + (s_1-l)^2 - 2(s_1-l)(s_2-l)\cos 2\alpha - 2x_0(s_1-s_2)\cos\alpha + x_0^2}$$

$$r_{32} = \sqrt{[(s_1-l) - x_0\cos\alpha - (s_2-l)\cos 2\alpha]^2 + [x_0\sin\alpha + (s_2-l)\sin 2\alpha]^2}$$

$$r'_{34} = \sqrt{(2l-s_1)^2 + (2l+s_2)^2 - 2(2l-s_1)(2l+s_2)\cos 2\alpha}$$

$$r_{34} = \sqrt{(2l-s_1)^2 + (2l+s_2)^2 - 2(2l-s_1)(2l+s_2)\cos 2\alpha + x_0^2 - 2x_0(s_2+s_1)\cos\alpha}$$

$$r_{34} = \sqrt{[(s_1-2l)^2 - x_0\cos\alpha + (s_2+2l)\cos 2\alpha]^2 + [x_0\sin\alpha - (s_2+2l)\sin 2\alpha]^2}$$

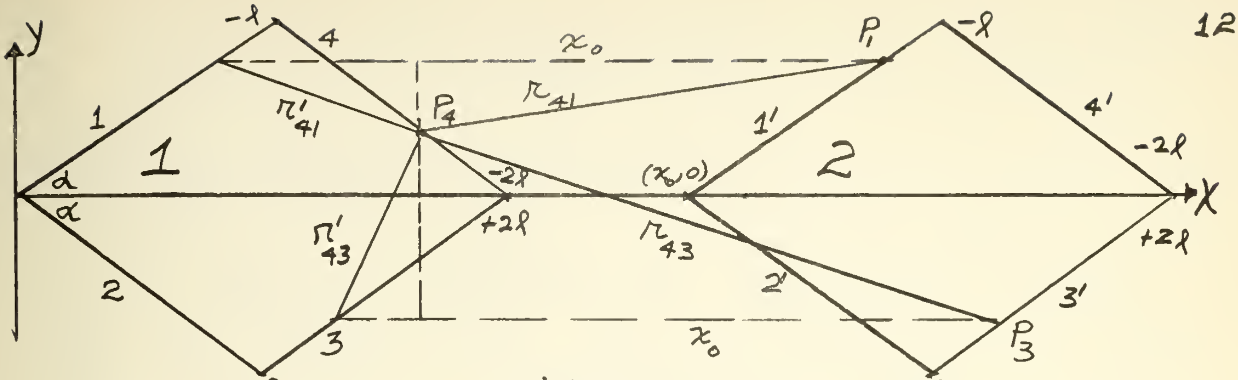

Figure 4

s_1 varies along rhombic 1, s_2 varies along rhombic 2

$-2l \le s_1 \le -l$

$-l \le s_2 \le 0$

$+l \le s_2 \le +2l$

Distance r_{41} is for paths $(4,1')$

r_{43} is for paths $(4,3')$

$$r'_{41} = \sqrt{(l+s_1)^2 + (l+s_2)^2 - 2(l+s_1)(l+s_2)\cos 2\alpha}$$

$$r_{41} = \sqrt{(s_1+l)^2 + (s_2+l)^2 - 2(l+s_1)(l+s_2)\cos 2\alpha + x_0^2 + 2x_0(s_1-s_2)\cos\alpha}$$

$$r_{41} = \sqrt{[(s_1+l) + x_0\cos\alpha - (s_2+l)\cos 2\alpha]^2 + [x_0\sin\alpha - (s_2+l)\sin 2\alpha]^2}$$

$$r'_{43} = \sqrt{(2l+s_1)^2 + (2l-s_2)^2 - 2(2l+s_1)(2l-s_2)\cos 2\alpha}$$

$$r_{43} = \sqrt{(s_1+2l)^2 + (s_2-2l)^2 + 2(s_1+2l)(s_2-2l)\cos 2\alpha + x_0^2 + 2x_0(s_1+s_2)\cos\alpha}$$

$$r_{43} = \sqrt{[(s_1+2l) + x_0\cos\alpha + (s_2-2l)\cos 2\alpha]^2 + [x_0\sin\alpha + (s_2-2l)\sin 2\alpha]^2}$$

$r_{41}(s_1,s_2) = r_{14}(s_1,s_2)$ $\qquad$ $r_{32}(s_1,s_2) = r_{23}(s_1,s_2)$

$r_{21}(s_1,s_2) = r_{12}(-s_1,-s_2)$ $\qquad$ $r_{43}(s_1,s_2) = r_{34}(-s_1,-s_1)$

APPENDIX B

Given

$$jKZ_{12}/30 = \oint_1\oint_2 e(r_{12})\left(\frac{\partial^2}{\partial s_2 \partial s_1} - K^2\cos\theta_{12}\right) g(Ks_1, Ks_2)\, ds_1 ds_2 \quad (1)$$

with $g(Ks_1, Ks_2) = R_e[f_1(P_1)^* f_2(P_2)]$, $\cos\theta_{12} ds_1 ds_2 = d\bar{r}_1 \cdot d\bar{r}_2$.

After postulating travelling waves of current along each rhombic antenna and considering the directions of the various legs of each antenna,

$$
\begin{aligned}
Z_{12}/jK60\sin^2\alpha = &\int_{s_1=-l}^{0}\int_{s_2=0}^{l} \cos K(s_1+s_2)\, e(r_{12})\, ds_2\, ds_1 \\
&- \int_{s_1=-l}^{0}\int_{s_2=-2l}^{-l} \cos K(s_1-s_2)\, e(r_{14})\, ds_2\, ds_1 \\
&+ \int_{s_1=0}^{l}\int_{s_2=-l}^{0} \cos K(s_1+s_2)\, e(r_{21})\, ds_2\, ds_1 \\
&- \int_{s_1=0}^{l}\int_{s_2=l}^{2l} \cos K(s_1-s_2)\, e(r_{23})\, ds_2\, ds_1 \\
&- \int_{s_1=l}^{2l}\int_{s_2=0}^{l} \cos K(s_1-s_2)\, e(r_{32})\, ds_2\, ds_1 \\
&+ \int_{s_1=l}^{2l}\int_{s_2=-2l}^{-l} \cos K(s_1+s_2)\, e(r_{34})\, ds_2\, ds_1 \\
&+ \int_{s_1=-2l}^{-l}\int_{s_2=l}^{2l} \cos K(s_1+s_2)\, e(r_{43})\, ds_2\, ds_1 \\
&- \int_{s_1=-2l}^{-l}\int_{s_2=-l}^{0} \cos K(s_1-s_2)\, e(r_{41})\, ds_2\, ds_1
\end{aligned}
\quad (2)
$$

Since $r_{21}(s_1, s_2) = r_{12}(-s_1, -s_2)$, $r_{43}(s_1, s_2) = r_{34}(-s_1, -s_2)$

$r_{41}(s_1, s_2) = r_{14}(s_1, s_2)$, $r_{32}(s_1, s_2) = r_{23}(s_1, s_2)$

with

$$r_{14} = \sqrt{[(s_1+l) + x_0\cos\alpha - (s_2+l)\cos 2\alpha]^2 + [x_0\sin\alpha - (s_2+l)\sin 2\alpha]^2}$$

$$r_{23} = \sqrt{[(s_1-l) - x_0\cos\alpha - (s_2-l)\cos 2\alpha]^2 + [x_0\sin\alpha + (s_2-l)\sin 2\alpha]^2}$$

$$r_{34} = \sqrt{[(s_1-2l) - x_0\cos\alpha + (s_2+2l)\cos 2\alpha]^2 + [x_0\sin\alpha - (s_2+2l)\sin 2\alpha]^2}$$

$$r_{12} = \sqrt{(s_1 + x_0\cos\alpha + s_2\cos 2\alpha)^2 + (x_0\sin\alpha + s_2\sin 2\alpha)^2},$$

equation (2) becomes,

$$\begin{aligned} Z_{12}/j120k\sin^2\alpha = &\int_{s_1=-l}^{0}\int_{s_2=0}^{l} \cos k(s_1+s_2)\, e(r_{12})\, ds_2\, ds_1 \\ &+ \int_{s_1=l}^{2l}\int_{s_2=-2l}^{-l} \cos k(s_1+s_2)\, e(r_{34})\, ds_2\, ds_1 \\ &-\left(\int_{s_1=-l}^{0}\int_{s_2=-2l}^{-l} + \int_{s_1=-2l}^{-l}\int_{s_2=-l}^{0}\right)\cos k(s_1-s_2)\, e(r_{14})\, ds_2\, ds_1 \\ &-\left(\int_{s_1=0}^{l}\int_{s_2=l}^{2l} + \int_{s_1=l}^{2l}\int_{s_2=0}^{l}\right)\cos k(s_1-s_2)\, e(r_{23})\, ds_2\, ds_1 \end{aligned} \tag{3}$$

In the first integral of equation (3), let $s_1 = -x_1$, $s_2 = x_2$; $r_{12} = \sqrt{(x_1 - x_0\cos\alpha - x_2\cos 2\alpha)^2 + (x_0\sin\alpha + x_2\sin 2\alpha)^2}$

In the second, let $s_1 = -x_1 + 2l$, $s_2 = x_3 - 2l$;

$r_{34} \sim r_{13} = \sqrt{(x_1 + x_0 - x_3\cos 2\alpha)^2 + (x_0\sin\alpha - x_3\sin 2\alpha)^2}$

In the first of the third, let $s_1 = x_1 - l$, $s_2 = -x_4 - l$;

$r_{14} = \sqrt{(x_1 + x_0\cos\alpha + x_4\cos 2\alpha)^2 + (x_0\sin\alpha + x_4\sin 2\alpha)^2}$

In the second of the third, let $s_1 = -x_1 - l$, $s_2 = x_5 - l$;

$r_{14} \sim r_{15} = \sqrt{(x_1 - x_0\cos\alpha + x_5\cos 2\alpha)^2 + (x_1\sin\alpha - x_5\sin 2\alpha)^2}$

In the first of the fourth, let $s_1 = -x_1 + l$, $s_2 = x_4 + l$;

$r_{23} \sim r_{14}$

In the second of the fourth, let $s_1 = x_1 + l$, $s_2 = -x_5 + l$;

$r_{23} \sim r_{15}$

Equation (3) then becomes,

$$Z_{12}/jk120\sin^2\alpha = \int_0^l\int_0^l \cos k(x_1-x_2)e(r_{12})\,dx_1dx_2 + \int_0^l\int_0^l \cos k(x_1-x_3)e(r_{13})\,dx_1dx_3$$
$$- \int_0^l\int_0^l \cos k(x_1+x_4)e(r_{14})\,dx_1dx_4 - \int_0^l\int_0^l \cos k(x_1+x_5)e(r_{15})\,dx_1dx_5 \qquad (4)$$

Changing to the exponential form,

$$Z_{12} = -jk60\sin^2\alpha\,[I_1+I_2+I_3+I_4-I_5-I_6-I_7-I_8] \qquad (5)$$

with

$$I_1 = \int_0^l\int_0^l \exp[-jk(x_5+x_1+r_{15})]\,r_{15}^{-1}\,dx_5dx_1$$
$$I_2 = \int_0^l\int_0^l \exp[jk(x_5+x_1-r_{15})]\,r_{15}^{-1}\,dx_5dx_1$$
$$I_3 = \int_0^l\int_0^l \exp[-jk(x_4+x_1+r_{14})]\,r_{14}^{-1}\,dx_4dx_1$$
$$I_4 = \int_0^l\int_0^l \exp[jk(x_4+x_1-r_{14})]\,r_{14}^{-1}\,dx_4dx_1$$
$$I_5 = \int_0^l\int_0^l \exp[jk(x_3-x_1-r_{13})]\,r_{13}^{-1}\,dx_3dx_1$$
$$I_6 = \int_0^l\int_0^l \exp[-jk(x_3-x_1+r_{13})]\,r_{13}^{-1}\,dx_3dx_1$$
$$I_7 = \int_0^l\int_0^l \exp[jk(x_2-x_1-r_{12})]\,r_{12}^{-1}\,dx_2dx_1$$
$$I_8 = \int_0^l\int_0^l \exp[-jk(x_2-x_1+r_{12})]\,r_{12}^{-1}\,dx_2dx_1$$

$$r_{12} = \sqrt{x_1^2+x_2^2+x_0^2-2x_0(x_1-x_2)\cos\alpha-2x_1x_2\cos 2\alpha}$$
$$r_{13} = \sqrt{x_1^2+x_3^2+x_0^2+2x_0(x_1-x_3)\cos\alpha-2x_1x_3\cos 2\alpha}$$
$$r_{14} = \sqrt{x_1^2+x_4^2+x_0^2+2x_0(x_1+x_4)\cos\alpha+2x_1x_4\cos 2\alpha}$$
$$r_{15} = \sqrt{x_1^2+x_5^2+x_0^2-2x_0(x_1+x_5)\cos\alpha+2x_1x_5\cos 2\alpha}$$

In I_8, let $x_2 \to x_1$, $x_1 \to x_3$; in I_6, let $x_3 \to x_1$, $x_1 \to x_2$; then

$$Z_{12} = -j60k\sin^2\alpha\,[I_1+I_2+I_3+I_4-2I_5-2I_7]$$

In carrying out the integrations, the following limits are encountered:

$r_{00} = x_0$

$r_{12}\big]_{0,l} = \sqrt{x_0^2 + l^2 + 2lx_0\cos\alpha}\,, \quad r_{12}\big]_{l,0} = \sqrt{x_0^2 + l^2 - 2lx_0\cos\alpha}$

$r_{13}\big]_{0,l} = \sqrt{x_0^2 + l^2 - 2lx_0\cos\alpha}\,, \quad r_{13}\big]_{l,0} = \sqrt{x_0^2 + l^2 + 2lx_0\cos\alpha}$

$r_{14}\big]_{0,l} = \sqrt{x_0^2 + l^2 + 2lx_0\cos\alpha} = r_{14}\big]_{l,0} = r_{12}\big]_{0,l} = r_{13}\big]_{l,0}$

$r_{15}\big]_{0,l} = \sqrt{x_0^2 + l^2 - 2lx_0\cos\alpha} = r_{15}\big]_{l,0} = r_{13}\big]_{0,l} = r_{12}\big]_{l,0}$

$r_{12}\big]_{l,l} = \sqrt{x_0^2 + (2l\sin\alpha)^2} = r_{13}\big]_{l,l}$

$r_{14}\big]_{l,l} = \sqrt{x_0^2 + (2l\cos\alpha)^2 + 4lx_0\cos\alpha} = x_0 + 2l\cos\alpha$

$r_{15}\big]_{l,l} = \sqrt{x_0^2 + (2l\cos\alpha)^2 - 4lx_0\cos\alpha} = x_0 - 2l\cos\alpha$

APPENDIX C

Given $$I_1 = \int_0^l \int_0^l \frac{e^{-jk(x_5+x_1+r_{15})}}{r_{15}} \, dx_5 \, dx_1 \qquad (1)$$

$$r_{15} = \sqrt{(x_1 - x_0\cos\alpha + x_5\cos 2\alpha)^2 + (x_0\sin\alpha - x_5\sin 2\alpha)^2}$$
$$= \sqrt{x_1^2 + x_5^2 + x_0^2 - 2x_0(x_1+x_5)\cos\alpha + 2x_1x_5\cos 2\alpha}$$

Let $m_1t = x_1 - x_0\cos\alpha + x_5\cos 2\alpha + r_{15}$ $\qquad m_1 = x_0\sin\alpha - x_5\sin 2\alpha$

$$m_1t_1 = -x_0\cos\alpha + x_5\cos 2\alpha + r_{05} \qquad r_{05} = \sqrt{x_5^2 + x_0^2 - 2x_0x_5\cos\alpha}$$

$$m_1t_2 = l - x_0\cos\alpha + x_5\cos 2\alpha + r_{l5} \qquad = \sqrt{(x_5 - x_0\cos\alpha)^2 + (x_0\sin\alpha)^2}$$

$$r_{l5} = \sqrt{(x_5 - x_0\cos\alpha + l\cos 2\alpha)^2 + (x_0\sin\alpha - l\sin 2\alpha)^2}$$

$$m_1t_1\Big]_0 = x_0(1-\cos\alpha) = 2x_0\sin^2\frac{\alpha}{2}$$

$$m_1t_1\Big]_l = l\cos 2\alpha - x_0\cos\alpha + r_{0l} \qquad r_{0l} = \sqrt{x_0^2 + l^2 - 2x_0 l\cos\alpha}$$

$$m_1t_2\Big]_0 = l - x_0\cos\alpha + r_{l0} \qquad r_{l0} = r_{0l}$$

$$m_1t_2\Big]_l = l(1+\cos 2\alpha) - x_0\cos\alpha + r_{ll} \qquad r_{ll} = x_0 - 2l\cos\alpha$$
$$= (x_0 - 2l\cos\alpha)(1-\cos\alpha)$$

$$m_1 dt = \left(1 + \frac{x_1 - x_0\cos\alpha + x_5\cos 2\alpha}{r_{15}}\right) dx_1 = \frac{m_1 t\, dx_1}{r_{15}}, \quad dx_1 = \frac{r_{15}}{t}\, dt$$

$$\frac{\partial(m_1t)}{\partial x_5} = \cos 2\alpha + \frac{x_5 - x_0\cos\alpha + x_1\cos 2\alpha}{r_{15}} = \frac{x_5 - x_0\cos\alpha + (x_1 + r_{15})\cos 2\alpha}{r_{15}}$$

$$\frac{1}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha + r_{05}\cos 2\alpha}{r_{05}(-x_0\cos\alpha + x_5\cos 2\alpha + r_{05})}$$

$$\frac{1}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha + (l + r_{l5})\cos 2\alpha}{r_{l5}(l - x_0\cos\alpha + x_5\cos 2\alpha + r_{l5})}$$

$$x_5 + x_1 + r_{15} = m_1 t + x_0\cos\alpha + 2x_5\sin^2\alpha$$

$$I_1 = e^{-jkx_0\cos\alpha} \int_0^l e^{-jk2x_5\sin^2\alpha} \int_{km_1t_1}^{km_1t_2} \frac{e^{-jt}}{t}\, dt \qquad (2)$$

$$I_1 = \frac{e^{-jkx_0\cos\alpha}}{-j2k\sin^2\alpha}\left[e^{-j2kx_5\sin^2\alpha}\left\{(Ci\,km_1t_2 - Ci\,km_1t_1) - j(Si\,km_1t_2 - Si\,km_1t_1)\right\}\right]_0^l \quad (3)$$

$$+\frac{e^{-jkx_0\cos\alpha}}{j2k\sin^2\alpha}\int_0^l e^{-j2kx_5\sin^2\alpha}\left\{\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial}{\partial x_5}(m_1t_2) - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial}{\partial x_5}(m_1t_1)\right\}dx_5$$

Let

$$A_{11} = e^{-jkx_0\cos\alpha}\int_0^l e^{-j2kx_5\sin^2\alpha}\left\{\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial}{\partial x_5}(m_1t_2) - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial}{\partial x_5}(m_1t_1)\right\}dx_5 \quad (4)$$

$$= \int_0^l \frac{e^{-jk(r_{l5}+l+x_5)}[x_5 - x_0\cos\alpha + (l+r_{l5})\cos 2\alpha]}{r_{l5}[l - x_0\cos\alpha + x_5\cos 2\alpha + r_{l5}]}dx_5$$

$$-\int_0^l \frac{e^{-jk(x_5+r_{05})}[x_5 - x_0\cos\alpha + r_{05}\cos 2\alpha]}{r_{05}[-x_0\cos\alpha + x_5\cos 2\alpha + r_{05}]}dx_5 \quad (5)$$

$$A_{12} = -\int_0^l \frac{e^{-jk(x_5+r_{05})}\{x_5 - x_0\cos\alpha + r_{05}\cos 2\alpha\}}{r_{05}[-x_0\cos\alpha + x_5\cos 2\alpha + r_{05}]}dx_5, \quad r_{05} = \sqrt{(x_5 - x_0\cos\alpha)^2 + (x_0\sin\alpha)^2} \quad (6)$$

Let

$$m_2 y = r_{05} + (x_5 - x_0\cos\alpha), \qquad m_2 = x_0\sin\alpha$$

$$m_2 y_1 = r_{00} - x_0\cos\alpha = x_0(1-\cos\alpha) = m_1t_1]_0$$

$$m_2 y_2 = r_{0l} + l - x_0\cos\alpha = m_1t_2]_0, \quad x_5 + r_{05} = m_2 y + x_0\cos\alpha$$

$$m_2\,dy = \left[1 + \frac{x_5 - x_0\cos\alpha}{r_{05}}\right]dx_5 = \frac{m_2 y\,dx_5}{r_{05}}, \quad dx_5 = \frac{r_{05}}{y}dy$$

$$\frac{m_2}{y} = \frac{(x_0\sin\alpha)^2}{r_{05} + (x_5 - x_0\cos\alpha)} = r_{05} - x_5 + x_0\cos\alpha$$

$$r_{05} = \frac{m_2}{2y}(y^2 + 1)$$

$$x_5 = m_2 y + x_0\cos\alpha - r_{05} = \frac{m_2}{2y}(y^2 - 1) + x_0\cos\alpha$$

$$x_5 = -m_2/y + x_0\cos\alpha + r_{05}$$

$$x_0\cos\alpha + x_5 - x_5\cos 2\alpha + l - x_0\cos\alpha + x_5\cos 2\alpha + r_{l5} = l + x_5 + r_{l5}$$

$$x_0\cos\alpha + x_5 - x_5\cos 2\alpha + x_5\cos 2\alpha + r_{05} = x_5 + r_{05}$$

$$A_{12} = -e^{-jKx_0\cos\alpha}\int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right] dy}{\left[2x_0\cos\alpha\sin^2\alpha + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + \frac{m_2}{2y}(y^2+1)\right]}$$

$$= -e^{-jKx_0\cos\alpha}\int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \, \frac{2y^2\cos^2\alpha - 2\sin^2\alpha}{-4y\sin\alpha\cos\alpha + 2y^2\cos^2\alpha + 2\sin^2\alpha}\, dy$$

$$= -e^{-jKx_0\cos\alpha}\int_{y_1}^{y_2} e^{-jKm_2 y}\frac{(y\cos\alpha - \sin\alpha)(y\cos\alpha + \sin\alpha)}{y(y\cos\alpha - \sin\alpha)^2}\, dy$$

$$= -e^{-jKx_0\cos\alpha}\int_{y_1}^{y_2} e^{-jKm_2 y}\frac{(y + \tan\alpha)}{y(y - \tan\alpha)}\, dy$$

$$A_{12} = -e^{-jKx_0\cos\alpha}\int_{y_1}^{y_2} e^{-jKm_2 y}\left(\frac{2}{y-\tan\alpha} - \frac{1}{y}\right) dy \qquad (7)$$

Let

$$-u = Km_2(y - \tan\alpha) = Km_2 y - Kx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$-jKm_2 y = ju - jKx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$u_1 = -K(m_2 y_1 - x_0\sin\alpha\tan\alpha) = -Kx_0\left(1 - \cos\alpha - \frac{\sin^2\alpha}{\cos\alpha}\right)$$

$$= Kx_0(\sec\alpha - 1)$$

$$u_2 = -K(m_2 y_2 - x_0\sin\alpha\tan\alpha) = -K\left[r_{0l} + l - x_0\left(\cos\alpha + \frac{\sin^2\alpha}{\cos\alpha}\right)\right]$$

$$= K[x_0\sec\alpha - (r_{0l} + l)]$$

Let $u = Km_2 y,\ u_1 = Km_2 y_1 = Kx_0(1-\cos\alpha)$

$$u_2 = Km_2 y_2 = K(r_{0l} + l - x_0\cos\alpha)$$

Then

$$\begin{aligned} A_{12} = & -2e^{-jKx_0\sec\alpha}\Big\{\mathrm{Ci}\,K[x_0\sec\alpha - (r_{0l}+l)] - \mathrm{Ci}\,K[x_0(\sec\alpha - 1)] \\ & \quad + j\big[\mathrm{Si}\,K[x_0\sec\alpha - (r_{0l}+l)] - \mathrm{Si}\,K[x_0(\sec\alpha-1)]\big]\Big\} \\ & + e^{-jKx_0\cos\alpha}\Big\{\mathrm{Ci}\,K[r_{0l}+l - x_0\cos\alpha] - \mathrm{Ci}\,K[x_0(1-\cos\alpha)] \\ & \quad - j\big[\mathrm{Si}\,K[r_{0l}+l - x_0\cos\alpha] - \mathrm{Si}\,K[x_0(1-\cos\alpha)]\big]\Big\} \end{aligned} \qquad (8)$$

$$A_{13}=\int_0^{l} e^{-jk(r_{l5}+l+x_5)}\frac{[x_5-x_0\cos\alpha+(l+r_{l5})\cos 2\alpha]}{r_{l5}\,[l-x_0\cos\alpha+x_5\cos 2\alpha+r_{l5}]}\,dx_5 - \quad (9)$$

$$r_{l5}=\sqrt{(x_5-x_0\cos\alpha+l\cos 2\alpha)^2+(x_0\sin\alpha-l\sin 2\alpha)^2}$$

$$m_2 y = r_{l5}+x_5-x_0\cos\alpha+l\cos 2\alpha\,,\qquad m_2 = x_0\sin\alpha-l\sin 2\alpha$$

$$m_2 y_1 = r_{l0}+l\cos 2\alpha-x_0\cos\alpha\,,\qquad r_{l0}=\sqrt{x_0^2+l^2-2x_0 l\cos\alpha}$$

$$m_2 y_2 = r_{ll}+2l\cos^2\alpha-x_0\cos\alpha\,,\qquad r_{ll}=\sqrt{x_0^2+(2l\cos\alpha)^2-4x_0 l\cos\alpha}$$

$$m_2\,dy=\left[1+\frac{x_5-x_0\cos\alpha+l\cos 2\alpha}{r_{l5}}\right]dx_5=\frac{m_2 y}{r_{l5}}dx_5\,,\quad dx_5=\frac{r_{l5}}{y}dy$$

$$r_{l5}+x_5+l=m_2 y+x_0\cos\alpha+2l\sin^2\alpha$$

$$\frac{m_2}{y}=\frac{(x_0\sin\alpha-l\sin 2\alpha)^2}{r_{l5}+[x_5-(x_0\cos\alpha-l\cos 2\alpha)]}=r_{l5}-[x_5-(x_0\cos\alpha-l\cos 2\alpha)]$$

$$r_{l5}=\frac{m_2}{2y}(y^2+1)$$

$$x_5=m_2 y-r_{l5}+x_0\cos\alpha-l\cos 2\alpha$$

$$x_5=-\frac{m_2}{y}+r_{l5}+x_0\cos\alpha-l\cos 2\alpha$$

$$=\frac{m_2}{2y}(y^2-1)+(x_0\cos\alpha-l\cos 2\alpha)$$

$$\boxed{\begin{aligned}&(x_0\cos\alpha-l\cos 2\alpha)\cos 2\alpha=\\&x_0\cos\alpha-2x_0\cos\alpha\sin^2\alpha-l\cos^2 2\alpha=\\&l-x_0\cos\alpha+(x_0\cos\alpha-l\cos 2\alpha)\cos 2\alpha=\\&l\sin^2 2\alpha-x_0\sin\alpha\sin 2\alpha\end{aligned}}$$

$$A_{13}=e^{-jk(x_0\cos\alpha+2l\sin^2\alpha)}\int_{y_1}^{y_2}\frac{e^{-jkm_2 y}}{y}\,\frac{\frac{m_2}{2y}(y^2-1)+\frac{m_2}{2y}(y^2+1)\cos 2\alpha}{\frac{m_2}{2y}(y^2+1)+\frac{m_2}{2y}(y^2-1)\cos 2\alpha-m_2\sin 2\alpha}\,dy$$

$$=e^{-jk(x_0\cos\alpha+2l\sin^2\alpha)}\int_{y_1}^{y_2}\frac{e^{-jkm_2 y}}{y}\,\frac{y^2\cos^2\alpha-\sin^2\alpha}{y^2\cos^2\alpha-2y\sin\alpha\cos\alpha+\sin^2\alpha}\,dy$$

$$=e^{-j(x_0\cos\alpha+2l\sin^2\alpha)}\int_{y_1}^{y_2}e^{-jkm_2 y}\left(\frac{2}{y-\tan\alpha}-\frac{1}{y}\right)dy \qquad (10)$$

Let

$$u=-km_2(y-\tan\alpha),\ -km_2 y=u-km_2\tan\alpha=u-kx_0\frac{\sin^2\alpha}{\cos\alpha}+2kl\sin^2\alpha$$

$$u_2=-k(r_{ll}+2l-x_0\sec\alpha),\quad u_1=-k(r_{l0}+l-x_0\sec\alpha)$$

Let

$$u=km_2 y$$

$$u_2=k(r_{ll}+2l\cos^2\alpha-x_0\cos\alpha)$$

$$u_1=k(r_{l0}+l\cos 2\alpha-x_0\cos\alpha)$$

$$
\begin{aligned}
A_{13} = 2e^{-jkx_0\sec\alpha} & \Big\{ \mathrm{Ci}\,k\,|r_{ll}+2l-x_0\sec\alpha| - \mathrm{Ci}\,k\,|r_{l0}+l-x_0\sec\alpha| \\
& -j\big[\mathrm{Si}\,k(r_{ll}+2l-x_0\sec\alpha) - \mathrm{Si}\,k(r_{l0}+l-x_0\sec\alpha)\big]\Big\} \\
- e^{-jk(x_0\cos\alpha+2l\sin^2\alpha)} & \Big\{\mathrm{Ci}\,k(r_{ll}+2l\cos^2\alpha-x_0\cos\alpha) - \mathrm{Ci}\,k(r_{l0}+l\cos 2\alpha-x_0\cos\alpha) \qquad (11)\\
& -j\big[\mathrm{Si}\,k(r_{ll}+2l\cos^2\alpha-x_0\cos\alpha) - \mathrm{Si}\,k(r_{l0}+l\cos 2\alpha-x_0\cos\alpha)\big]\Big\}
\end{aligned}
$$

Substituting into equation (3),

$$
\begin{aligned}
& -jI_1 2k\sin^2\alpha = \\
& 2e^{-jkx_0\sec\alpha}\Big\{\mathrm{Ci}\,k(x_0\sec\alpha-l-r_{0l}) - \mathrm{Ci}\,k[x_0(\sec\alpha-1)] \\
& \qquad - \mathrm{Ci}\,k\,|r_{ll}+2l-x_0\sec\alpha| + \mathrm{Ci}\,k\,|r_{l0}+l-x_0\sec\alpha| \\
& +j\big[\mathrm{Si}\,k(x_0\sec\alpha-l-r_{0l}) - \mathrm{Si}\,k[x_0(\sec\alpha-1)] + \mathrm{Si}\,k(r_{ll}+2l-x_0\sec\alpha) \\
& \quad - \mathrm{Si}\,k(r_{l0}+l-x_0\sec\alpha)\big]\Big\} \qquad (12) \\
& -2e^{-jkx_0\cos\alpha}\Big\{\mathrm{Ci}\,k(r_{0l}+l-x_0\cos\alpha) - \mathrm{Ci}\,k[x_0(1-\cos\alpha)] \\
& \qquad -j\big[\mathrm{Si}\,k(r_{0l}+l-x_0\cos\alpha) - \mathrm{Si}\,k[x_0(1-\cos\alpha)]\big]\Big\} \\
& +2e^{-jk(x_0\cos\alpha+2l\sin^2\alpha)}\Big\{\mathrm{Ci}\,k(r_{ll}+2l\cos^2\alpha-x_0\cos\alpha) \\
& \quad - \mathrm{Ci}\,k(r_{10}+l\cos 2\alpha-x_0\cos\alpha) - j\big[\mathrm{Si}\,k(r_{ll}+2l\cos^2\alpha-x_0\cos\alpha) \\
& \quad - \mathrm{Si}\,k(r_{l0}+l\cos 2\alpha-x_0\cos\alpha)\big]\Big\}
\end{aligned}
$$

upon breaking equation (12) into its real and imaginary components,

$$-ik\sin^2\alpha\, I_1 =$$

$$\cos(kx_0\sec\alpha)\Big\{2\,\mathrm{Ci}\,k\big(x_0\sec\alpha-l-\sqrt{x_0^2+l^2-2lx_0\cos\alpha}\big)-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Ci}\,k[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]\Big\}$$
$$+\sin(kx_0\sec\alpha)\Big\{2\,\mathrm{Si}\,k\big(x_0\sec\alpha-l-\sqrt{x_0^2+l^2-2lx_0\cos\alpha}\big)-\mathrm{Si}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Si}\,k[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]\Big\}$$
$$-\cos(kx_0\cos\alpha)\Big\{\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l-x_0\cos\alpha\big)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\sin(kx_0\cos\alpha)\Big\{\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l-x_0\cos\alpha\big)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\cos k(x_0\cos\alpha+2l\sin^2\alpha)\Big\{\mathrm{Ci}\,k[(x_0-2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l\cos2\alpha-x_0\cos\alpha\big)\Big\}$$
$$-\sin k(x_0\cos\alpha+2l\sin^2\alpha)\Big\{\mathrm{Si}\,k[(x_0-2l\cos\alpha)(1-\cos\alpha)$$
$$-\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l\cos2\alpha-x_0\cos\alpha\big)\Big\}$$

$$+j\Big[\cos k(x_0\sec\alpha)\Big\{2\,\mathrm{Si}\,k\big(x_0\sec\alpha-l-\sqrt{x_0^2+l^2-2lx_0\cos\alpha}\big)-\mathrm{Si}\,k[x_0(\sec\alpha-1)] \qquad (13)$$
$$-\mathrm{Si}\,k[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]\Big\}$$
$$-\sin k(x_0\sec\alpha)\Big\{2\,\mathrm{Ci}\,k\big(x_0\sec\alpha-l-\sqrt{x_0^2+l^2-2lx_0\cos\alpha}\big)-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Ci}\,k[x_0(\sec\alpha-1)-2l(1-\cos\alpha)]\Big\}$$
$$+\cos k(x_0\cos\alpha)\Big\{\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l-x_0\cos\alpha\big)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\sin k(x_0\cos\alpha)\Big\{\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l-x_0\cos\alpha\big)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$-\cos k(x_0\cos\alpha+2l\sin^2\alpha)\Big\{\mathrm{Si}\,k[(x_0-2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l\cos2\alpha-x_0\cos\alpha\big)\Big\}$$
$$-\sin k(x_0\cos\alpha+2l\sin^2\alpha)\Big\{\mathrm{Ci}\,k[(x_0-2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+l\cos2\alpha-x_0\cos\alpha\big)\Big\}\Big]$$

APPENDIX D

Given
$$I_2 = \int_0^l \int_0^l \frac{e^{-jK(r_{15}-x_1-x_5)}}{r_{15}}\, dx_5\, dx_1 \qquad (1)$$

$$r_{15} = \sqrt{(x_1 - x_0\cos\alpha + x_5\cos 2\alpha)^2 + (x_0\sin\alpha - x_5\sin 2\alpha)^2}$$

Let

$m_1 t = r_{15} - x_1 + x_0\cos\alpha - x_5\cos 2\alpha$, $\quad m_1 = x_0\sin\alpha - x_5\sin 2\alpha$

$m_1 t_1 = r_{05} + x_0\cos\alpha - x_5\cos 2\alpha$, $\quad r_{05} = \sqrt{(x_5 - x_0\cos\alpha)^2 + (x_0\sin\alpha)^2}$

$m_1 t_2 = r_{l5} - l + x_0\cos\alpha - x_5\cos 2\alpha$, $\quad r_{l5} = \sqrt{(x_5 - x_0\cos\alpha + l\cos 2\alpha)^2 + (x_0\sin\alpha - l\sin 2\alpha)^2}$

$m_1 t_1\big]_0 = x_0(1+\cos\alpha)$, $\quad r_{00} = x_0$

$m_1 t_1\big]_l = r_{0l} + x_0\cos\alpha - l\cos 2\alpha$, $\quad r_{0l} = r_{l0} = \sqrt{x_0^2 + l^2 - 2lx_0\cos\alpha}$

$m_1 t_2\big]_0 = r_{l0} - l + x_0\cos\alpha$, $\quad r_{ll} = \sqrt{2l^2 + x_0^2 - 4lx_0\cos\alpha + 2l^2\cos^2\alpha}$

$\qquad = x_0 - 2l\cos\alpha$

$m_1 t_2\big]_l = r_{ll} + x_0\cos\alpha - 2l\cos^2\alpha$,

$\qquad = (x_0 - 2l\cos\alpha)(1+\cos\alpha)$

$$m_1\, dt = \left(\frac{x_1 - x_0\cos\alpha + x_5\cos 2\alpha}{r_{15}} - 1\right)dx_1 = -\frac{m_1 t}{r_{15}}\,dx_1, \quad dx_1 = -\frac{r_{15}}{t}\,dt$$

$$\frac{\partial(m_1 t)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha + x_1\cos 2\alpha}{r_{15}} - \cos 2\alpha = \frac{x_5 - x_0\cos\alpha + (x_1 - r_{15})\cos 2\alpha}{r_{15}}$$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha - r_{05}\cos 2\alpha}{r_{05}(r_{05} + x_0\cos\alpha - x_5\cos 2\alpha)}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_5} = \frac{x_5 - x_0\cos\alpha + (l - r_{l5})\cos 2\alpha}{r_{l5}(r_{l5} - l + x_0\cos\alpha - x_5\cos 2\alpha)}$$

$$x_5 + x_1 - r_{15} = -m_1 t + 2x_5\sin^2\alpha + x_0\cos\alpha$$

$$I_2 = -e^{jKx_0\cos\alpha}\int_0^l e^{jK2x_5\sin^2\alpha}\int_{Km_1t_1}^{Km_1t_2} \frac{e^{-jt}}{t}\,dt \qquad (2)$$

$$= -\frac{e^{jKx_0\cos\alpha}}{j2K\sin^2\alpha}\left[e^{jK2x_5\sin^2\alpha}\left\{\mathrm{Ci}\,Km_1t_2 - \mathrm{Ci}\,Km_1t_1 - j(\mathrm{Si}\,Km_1t_2 - \mathrm{Si}\,Km_1t_1)\right\}\right]_0^l$$
$$+ \frac{e^{jKx_0\cos\alpha}}{j2K\sin^2\alpha}\int_0^l e^{jK2x_5\sin^2\alpha}\left[\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_5} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_5}\right]dx_5 \qquad (3)$$

Let
$$A_{21}=e^{jkx_0\cos\alpha}\int_0^{l} e^{-jk2x_5\sin^2\alpha}\left[\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_5}-\frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_5}\right]dx_5 \qquad (4)$$

$$x_0\cos\alpha+2x_5\sin^2\alpha-r_{05}=x_0\cos\alpha+x_5\cos 2\alpha=x_5-r_{05}$$

$$x_0\cos\alpha+2x_5\sin^2\alpha-r_{l5}+l-x_0\cos\alpha+x_5\cos 2\alpha=x_5+l-r_{l5}$$

$$A_{21}=\int_0^{l} e^{-jk(l+x_5-r_{l5})}\frac{[x_5-x_0\cos\alpha+(l-r_{l5})\cos 2\alpha]}{r_{l5}[r_{l5}-l+x_0\cos\alpha-x_5\cos 2\alpha]}dx_5$$

$$-\int_0^{l} e^{jk(x_5-r_{05})}\frac{[x_5-x_0\cos\alpha-r_{05}\cos 2\alpha]}{r_{05}[r_{05}+x_0\cos\alpha-x_5\cos 2\alpha]}dx_5 \qquad (5)$$

Set
$$A_{22}=-\int_0^{l} e^{-jk(x_5-r_{05})}\frac{[x_5-x_0\cos\alpha-r_{05}\cos 2\alpha]}{r_{05}[r_{05}+x_0\cos\alpha-x_5\cos 2\alpha]}dx_5 \qquad (6)$$

$$r_{05}=\sqrt{(x_5-x_0\cos\alpha)^2+(x_0\sin\alpha)^2}$$

$$m_2y=r_{05}-(x_5-x_0\cos\alpha)\ , \quad m_2=x_0\sin\alpha$$

$$m_2y_1=r_{00}+x_0\cos\alpha=x_0(1+\cos\alpha)=m_1t_1]_0$$

$$m_2y_2=r_{0l}-l+x_0\cos\alpha=m_1t_2]_0$$

$$m_2dy=\left[\frac{x_5-x_0\cos\alpha}{r_{05}}-1\right]dx_5=-\frac{m_2y}{r_{05}}dx_5\ ,\ dx_5=-\frac{r_{05}}{y}dy$$

$$x_5-r_{05}=-m_2y+x_0\cos\alpha \qquad x_0\cos\alpha(1-\cos 2\alpha)=2x_0\cos\alpha\sin^2\alpha$$

$$\frac{m_2}{y}=\frac{(x_0\sin\alpha)^2}{r_{05}-x_5+x_0\cos\alpha}=r_{05}+x_5-x_0\cos\alpha \qquad =x_0\sin\alpha\sin 2\alpha=m_2\sin 2\alpha$$

$$r_{05}=\frac{m_2}{2y}(y^2+1)\ , \quad x_5-x_0\cos\alpha=-\frac{m_2}{2y}(y^2-1)$$

$$A_{22}=e^{jkx_0\cos\alpha}\int_{y_1}^{y_2}\frac{e^{-jkm_2y}\left[-\frac{m_2}{2y}(y^2-1)-\frac{m_2}{2y}(y^2+1)\cos 2\alpha\right]dy}{y\left[\frac{m_2}{2y}(y^2+1)+\frac{m_2}{2y}(y^2-1)\cos 2\alpha+m_2\sin 2\alpha\right]} \qquad (7)$$

$$=-e^{jkx_0\cos\alpha}\int_{y_1}^{y_2}e^{-jkm_2y}\left(\frac{2}{y+\tan\alpha}-\frac{1}{y}\right)dy \qquad (8)$$

Let
$$u=km_2(y+\tan\alpha)=km_2y+kx_0\frac{\sin^2\alpha}{\cos\alpha}\ ,\ -jkm_2y=-ju+jkx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$u_2=k(r_{0l}-l+x_0\sec\alpha)\ , \quad u_1=kx_0(\sec\alpha+1)$$

$$u=km_2y\ ,\ u_2=k(r_{0l}-l+x_0\cos\alpha),\ u_1=kx_0(1+\cos\alpha)$$

$$A_{22} = -2e^{jkx_0 \sec\alpha}\Big\{ \mathrm{Ci}\,k(r_{0\ell} - \ell + x_0 \sec\alpha) - \mathrm{Ci}\,k[x_0(\sec\alpha + 1)] - j\big[\mathrm{Si}\,k(r_{0\ell} - \ell + x_0\sec\alpha) - \mathrm{Si}\,k[x_0(\sec\alpha+1)]\big]\Big\}$$

$$+ e^{jkx_0\cos\alpha}\Big\{\mathrm{Ci}\,k(r_{0\ell} - \ell + x_0\cos\alpha) - \mathrm{Ci}\,k[x_0(1+\cos\alpha)] - j\big[\mathrm{Si}\,k(r_{0\ell} - \ell + x_0\cos\alpha) - \mathrm{Si}\,k[x_0(1+\cos\alpha)]\big]\Big\} \qquad (9)$$

Set

$$A_{23} = \int_0^\ell e^{jk(\ell + x_5 - r_{\ell 5})} \frac{[x_5 - x_0\cos\alpha + (\ell - r_{\ell 5})\cos 2\alpha]}{r_{\ell 5}\,[r_{\ell 5} - \ell + x_0\cos\alpha - x_5\cos 2\alpha]}\, dx_5$$

$$r_{\ell 5} = \sqrt{(x_5 - x_0\cos\alpha + \ell\cos 2\alpha)^2 + (x_0\sin\alpha - \ell\sin 2\alpha)^2}$$

$$m_2 y = r_{\ell 5} - x_5 + x_0\cos\alpha - \ell\cos 2\alpha, \quad m_2 = x_0\sin\alpha - \ell\sin 2\alpha$$

$$m_2 y_2 = x_0(1+\cos\alpha) - \ell(1+\cos 2\alpha) - 2\ell\cos\alpha, \quad r_{\ell 0} = \sqrt{x_0^2 + \ell^2 - 2\ell x_0\cos\alpha}$$

$$= (x_0 - 2\ell\cos\alpha)(1+\cos\alpha) = m_1 t_2]_\ell, \quad r_{\ell\ell} = x_0 - 2\ell\cos\alpha$$

$$m_2 y_1 = r_{\ell 0} + x_0\cos\alpha - \ell\cos 2\alpha = m_1 t_1]_\ell$$

$$m_2\, dy = \left[\frac{x_5 - x_0\cos\alpha + \ell\cos 2\alpha}{r_{\ell 5}} - 1\right] dx_5 = -\frac{m_2 y}{r_{\ell 5}}\, dx_5, \quad dx_5 = -\frac{r_{\ell 5}\, dy}{y}$$

$$\ell + x_5 - r_{\ell 5} = -m_2 y + x_0\cos\alpha + 2\ell\sin^2\alpha$$

$$\frac{m_2}{y} = \frac{(x_0\sin\alpha - \ell\sin 2\alpha)^2}{r_{\ell 5} - (x_5 - x_0\cos\alpha + \ell\cos 2\alpha)} = r_{\ell 5} + x_5 - x_0\cos\alpha + \ell\cos 2\alpha$$

$$\boxed{\begin{aligned} &x_0\cos\alpha(1-\cos 2\alpha) - \ell(1-\cos^2 2\alpha) \\ &= 2x_0\cos\alpha\sin^2\alpha - \ell\sin^2 2\alpha \\ &= \sin 2\alpha(x_0\sin\alpha - \ell\sin 2\alpha) = m_2\sin 2\alpha \end{aligned}}$$

$$r_{\ell 5} = \frac{m_2}{2y}(y^2+1)$$

$$x_5 - x_0\cos\alpha + \ell\cos 2\alpha = -\frac{m_2}{2y}(y^2-1)$$

$$A_{23} = e^{jk(x_0\cos\alpha + 2\ell\sin^2\alpha)} \int_{y_1}^{y_2} \frac{e^{-jkm_2 y}\left[\frac{m_2}{2y}(y^2+1)\cos 2\alpha + \frac{m_2}{2y}(y^2-1)\right] dy}{y\left[\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + m_2\sin 2\alpha\right]} \qquad (10)$$

$$= e^{jk(x_0\cos\alpha + 2\ell\sin^2\alpha)} \int_{y_1}^{y_2} e^{-jkm_2 y}\left[\frac{2}{y + \tan\alpha} - \frac{1}{y}\right] dy \qquad (11)$$

Let

$$u = km_2(y + \tan\alpha) = k\left(m_2 y + x_0\frac{\sin^2\alpha}{\cos\alpha} - 2\ell\sin^2\alpha\right), \quad -km_2 y = -ku + k\left(x_0\frac{\sin^2\alpha}{\cos\alpha} - 2\ell\sin^2\alpha\right)$$

$$u_2 = k[x_0(\sec\alpha + 1) - 2\ell(1+\cos\alpha)], \quad u_1 = k[r_{\ell 0} + x_0\sec\alpha - \ell]$$

$$u = -km_2 y, \quad u_2 = k(x_0 - 2\ell\cos\alpha)(1+\cos\alpha) = m_1 t_2]_\ell, \quad u_1 = k(r_{\ell 0} + x_0\cos\alpha - \ell\cos 2\alpha) = m_1 t_1]_\ell$$

$$
\begin{aligned}
A_{23} = 2e^{jkx_0\sec\alpha}&\{Cik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)] - Cik(r_{l0}+x_0\sec\alpha-l) \\
&-j[Sik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)] - Sik(r_{l0}+x_0\sec\alpha-l)]\} \qquad (12)\\
-e^{jk(x_0\cos\alpha+2l\sin^2\alpha)}&\{Cik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Cik(r_{l0}+x_0\cos\alpha-l\cos2\alpha) \\
&-j[Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Sik(r_{l0}+x_0\sec\alpha-l)]\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-j2k\sin^2\alpha\, I_2 = {}& \\
e^{jkx_0\sec\alpha}&\{2Cik[r_{l0}+x_0\sec\alpha-l] - Cik[x_0(\sec\alpha+1)] - Cik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)] \\
&-j[2Sik[r_{l0}+x_0\sec\alpha-l] - Sik[x_0(\sec\alpha+1)] - Sik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]]\} \\
-e^{jkx_0\cos\alpha}&\{Cik[r_{0l}+x_0\cos\alpha-l] - Cik[x_0(1+\cos\alpha)] - j[Sik[r_{0l}+x_0\cos\alpha-l] - Sik[x_0(1+\cos\alpha)]]\} \qquad (13)\\
+e^{jk(x_0\cos\alpha+2l\sin^2\alpha)}&\{Cik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Cik(r_{l0}+x_0\cos\alpha-l\cos2\alpha) \\
&-j[Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Sik(r_{l0}+x_0\cos\alpha-l\cos2\alpha)]\}
\end{aligned}
$$

or

$$
\begin{aligned}
-jk\sin^2\alpha\, I_2 = {}& \\
\cos(kx_0\sec\alpha)&\{2Cik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\sec\alpha-l) - Cik[x_0(\sec\alpha+1)] \\
&- Cik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]\} \\
+\sin(kx_0\sec\alpha)&\{2Sik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\sec\alpha-l) - Sik[x_0(\sec\alpha+1)] \\
&- Sik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]\} \\
-\cos(kx_0\cos\alpha)&\{Cik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\cos\alpha-l) - Cik[x_0(1+\cos\alpha)]\} \\
-\sin(kx_0\cos\alpha)&\{Sik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\cos\alpha-l) - Sik[x_0(1+\cos\alpha)]\} \\
+\cos k(x_0\cos\alpha+2l\sin^2\alpha)&\{Cik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Cik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\cos\alpha-l\cos2\alpha)\} \\
+\sin k(x_0\cos\alpha+2l\sin^2\alpha)&\{Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Sik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\cos\alpha-l\cos2\alpha)\} \\
+j\Big[-\cos(kx_0\sec\alpha)&\{2Sik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\sec\alpha-l) - Sik[x_0(\sec\alpha+1)] \qquad (14)\\
&- Sik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]\} \\
+\sin(kx_0\sec\alpha)&\{2Cik(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}+x_0\sec\alpha-l) - Cik[x_0(\sec\alpha+1)] \\
&- Cik[x_0(\sec\alpha+1)-2l(1+\cos\alpha)]\} \\
+\cos(kx_0\cos\alpha)&\{Sik(\sqrt{x_0^2+l^2-2l\cos\alpha}+x_0\cos\alpha-l) - Sik[x_0(1+\cos\alpha)]\} \\
-\sin(kx_0\cos\alpha)&\{Cik(\sqrt{x_0^2+l^2-2l\cos\alpha}+x_0\cos\alpha-l) - Cik[x_0(1+\cos\alpha)]\} \\
-\cos k(x_0\cos\alpha+2l\sin^2\alpha)&\{Sik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Sik(\sqrt{x_0^2+l^2-2l\cos\alpha}+x_0\cos\alpha-l\cos2\alpha)\} \\
+\sin k(x_0\cos\alpha+2l\sin^2\alpha)&\{Cik[(x_0-2l\cos\alpha)(1+\cos\alpha)] - Cik(\sqrt{x_0^2+l^2-2l\cos\alpha}+x_0\cos\alpha-l\cos2\alpha)\}\Big]
\end{aligned}
$$

APPENDIX E

Given $I_3 = \int_0^l \int_0^l \exp[-jk(x_4 + x_1 + r_{14})]\, r_{14}^{-1}\, dx_4\, dx_1$ (1)

$r_{14} = \sqrt{(x_1 + x_0\cos\alpha + x_4\cos 2\alpha)^2 + (x_0\sin\alpha + x_4\sin 2\alpha)^2}$

Let

$m_1 t = x_1 + x_0\cos\alpha + x_4\cos 2\alpha + r_{14}$, $m_1 = x_0\sin\alpha + x_4\sin 2\alpha$

$m_1 t_1 = x_0\cos\alpha + x_4\cos 2\alpha + r_{04}$, $r_{04} = \sqrt{(x_4 + x_0\cos\alpha)^2 + (x_0\sin\alpha)^2}$

$m_1 t_2 = l + x_0\cos\alpha + x_4\cos 2\alpha + r_{l4}$, $r_{l4} = \sqrt{(x_4 + x_0\cos\alpha + l\cos 2\alpha)^2 + (x_0\sin\alpha + l\sin 2\alpha)^2}$

$m_1 t_1\big]_0 = x_0(1+\cos\alpha)$, $r_{l0} = r_{0l} = \sqrt{x_0^2 + l^2 + 2lx_0\cos\alpha}$

$m_1 t_1\big]_l = x_0\cos\alpha + l\cos 2\alpha + r_{0l}$, $r_{ll} = \sqrt{2l^2 + x_0^2 + 4lx_0\cos\alpha + 2l^2\cos 2\alpha}$

$m_1 t_2\big]_0 = l + x_0\cos\alpha + r_{l0}$, $\quad = x_0 + 2l\cos\alpha$

$r_{00} = x_0$

$m_1 t_2\big]_l = l(1+\cos 2\alpha) + x_0\cos\alpha + r_{ll} = (x_0 + 2l\cos\alpha)(1+\cos\alpha)$

$x_4 + x_1 + r_{14} = m_1 t - x_0\cos\alpha + 2x_4\sin^2\alpha$, $dx_1 = \frac{r_{14}}{t}\,dt$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_4} = \frac{x_4 + x_0\cos\alpha + r_{04}\cos 2\alpha}{r_{04}[x_0\cos\alpha + x_4\cos 2\alpha + r_{04}]}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_4} = \frac{x_4 + x_0\cos\alpha + (l + r_{l4})\cos 2\alpha}{r_{l4}[l + x_0\cos\alpha + x_4\cos 2\alpha + r_{l4}]}$$

and

$$I_3 = e^{jkx_0\cos\alpha}\int_0^l e^{-jk2x_4\sin^2\alpha}\int_{km_1t_1}^{km_1t_2}\frac{e^{-jt}}{t}\,dt \qquad (2)$$

$$= -\frac{e^{jkx_0\cos\alpha}}{j2k\sin^2\alpha}\left[e^{-j2kx_4\sin^2\alpha}\left\{\mathrm{Ci}\,km_1t_2 - \mathrm{Ci}\,km_1t_1 - j\left[\mathrm{Si}\,km_1t_2 - \mathrm{Si}\,km_1t_1\right]\right\}\right]_0^l$$

$$+\frac{e^{jkx_0\cos\alpha}}{j2k\sin^2\alpha}\int_0^l e^{-j2kx_4\sin^2\alpha}\left\{\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_4}\right\}dx_4 \qquad (3)$$

$x_0\cos\alpha - 2x_4\sin^2\alpha - l - x_0\cos\alpha - x_4\cos 2\alpha - r_{l4} = -(l + x_4 + r_{l4})$

$x_0\cos\alpha - 2x_4\sin^2\alpha - x_0\cos\alpha - x_4\cos 2\alpha - r_{04} = -(x_4 + r_{04})$

Set

$$A_{31} = \int_0^l e^{-jk(r_{l4}+l+x_4)}\frac{x_4 + x_0\cos\alpha + (l + r_{l4})\cos 2\alpha}{r_{l4}(l + x_0\cos\alpha + x_4\cos 2\alpha + r_{l4})}\,dx_4$$

$$-\int_0^l e^{-jk(x_4+r_{04})}\frac{x_4 + \cos\alpha + r_{04}\cos 2\alpha}{r_{04}(x_0\cos\alpha + x_4\cos 2\alpha + r_{04})}\,dx_4 \qquad (4)$$

Let
$$A_{32} = -\int_0^{l} e^{-jK(x_4+r_{04})} \frac{x_4 + x_0\cos\alpha + r_{04}\cos 2\alpha}{r_{04}(x_0\cos\alpha + x_4\cos 2\alpha + r_{04})}\, dx_4 \qquad (5)$$

$m_2 y = r_{04} + (x_4 + x_0\cos\alpha)\ , \quad m_2 = x_0 \sin\alpha$

$m_2 y_1 = r_{00} + x_0\cos\alpha = x_0(1+\cos\alpha) = m_1 t_1]_0$

$m_2 y_2 = r_{0l} + l + x_0\cos\alpha = m_1 t_2]_0\ , \quad dx_4 = \frac{r_{04}}{y}dy$

$x_4 + r_{04} = m_2 y - x_0\cos\alpha$

$\frac{m_2}{y} = \frac{(x_0\sin\alpha)^2}{r_{04}+x_4+x_0\cos\alpha} = r_{04} - x_4 - x_0\cos\alpha$

$r_{04} = \frac{m_2}{2y}(y^2+1)\ , \quad x_4 + x_0\cos\alpha = \frac{m_2}{2y}(y^2-1)$

$$A_{32} = -e^{jKx_0\cos\alpha}\int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right] dy}{\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + 2x_0\cos\alpha\sin^2\alpha}$$

$$= -e^{jKx_0\cos\alpha}\int_{y_1}^{y_2} e^{-jKm_2 y}\left(\frac{2}{y+\tan\alpha} - \frac{1}{y}\right) dy \qquad (6)$$

Let

$u = Km_2(y+\tan\alpha) = Km_2 y + Kx_0\frac{\sin^2\alpha}{\cos\alpha}\ , \quad -jKm_2 y = -ju + jKx_0\frac{\sin^2\alpha}{\cos\alpha}$

$u_1 = Km_2 y_1 + Kx_0\frac{\sin^2\alpha}{\cos\alpha} = Kx_0(1+\sec\alpha)\ , \quad u_2 = K(r_{0l} + l + x_0\sec\alpha)$

$u = Km_2 y\ , \quad u_1 = Kx_0(1+\cos\alpha)\ , \quad u_2 = K(r_{0l} + l + x_0\cos\alpha)$

$$A_{32} = -2e^{jKx_0\sec\alpha}\int_{Kx_0(1+\sec\alpha)}^{K(r_{0l}+l+x_0\sec\alpha)} \frac{e^{-ju}}{u}du + e^{jKx_0\cos\alpha}\int_{Kx_0(1+\cos\alpha)}^{K(r_{0l}+l+x_0\cos\alpha)} \frac{e^{-ju}}{u}du \qquad (7)$$

Let
$$A_{33} = \int_0^{l} e^{-jK(r_{l4}+l+x_4)} \frac{x_4 + x_0\cos\alpha + (l + r_{l4})\cos 2\alpha}{r_{l4}(l + x_0\cos\alpha + x_4\cos 2\alpha + r_{l4})}\, dx_4 \qquad (8)$$

$m_2 y = r_{l4} + x_4 + x_0\cos\alpha + l\cos 2\alpha\ , \quad m_2 = x_0\sin\alpha + l\sin 2\alpha$

$m_2 y_1 = r_{l0} + l\cos 2\alpha + x_0\cos\alpha = m_1 t_1]_l$

$m_2 y_2 = r_{ll} + 2l\cos^2\alpha = (x_0 + 2l\cos\alpha)(1+\cos\alpha) = m_1 t_2]_l$

$r_{l4} + l + x_4 = m_2 y - x_0\cos\alpha + 2l\sin^2\alpha \qquad dx_4 = \frac{r_{l4}}{y}dy$

$m_2/y = r_{l4} - (x_4 + x_0\cos\alpha + l\cos 2\alpha) \qquad l + x_0\cos\alpha - x_0\cos\alpha\cos 2\alpha - l\cos^2 2\alpha =$

$r_{l4} = \frac{m_2}{2y}(y^2+1) \qquad 2x_0\cos\alpha\sin^2\alpha + l\sin^2 2\alpha =$

$\qquad 2\sin\alpha\cos\alpha(x_0\sin\alpha + l\sin 2\alpha) = m_2\sin 2\alpha$

$x_4 + x_0\cos\alpha + l\cos 2\alpha = \frac{m_2}{2y}(y^2-1)$

$$A_{33} = e^{jK(x_0\cos\alpha - 2l\sin^2\alpha)}\int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right] dy}{\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + m_2\sin 2\alpha} \qquad (9)$$

$$A_{33}=e^{jK(x_0\cos\alpha-2l\sin^2\alpha)}\int_{y_1}^{y_2}e^{-jKm_2y}\left(\frac{2}{y+\tan\alpha}-\frac{1}{y}\right)dy \qquad (10)$$

$u=Km_2(y+\tan\alpha)$, $-jKm_2y=-ju+jK(x_0\frac{\sin^2\alpha}{\cos\alpha}+2l\sin^2\alpha)$

$u_1=K(x_0\sec\alpha+l+r_{l0})$, $u_2=K[x_0(1+\sec\alpha)+2l(1+\cos\alpha)]$

$u=Km_2y$, $u_1=K(r_{l0}+x_0\cos\alpha+l\cos2\alpha)$, $u_2=K(x_0+2l\cos\alpha)(1+\cos\alpha)$ (11)

$$A_{33}=2e^{jKx_0\sec\alpha}\int_{K(x_0\sec\alpha+l+r_{l0})}^{K[x_0(1+\sec\alpha)+2l(1+\cos\alpha)]}\frac{e^{-ju}}{u}du-e^{jK(x_0\cos\alpha-2l\sin^2\alpha)}\int_{K(r_{l0}+x_0\cos\alpha+l\cos2\alpha)}^{K(x_0+2l\cos\alpha)(1+\cos\alpha)}e^{-ju}/u\,du$$

Therefore

$$\begin{aligned}-j2K\sin^2\alpha\, I_3 &= 2e^{jKx_0\sec\alpha}\{2CiK(r_{0l}+l+x_0\sec\alpha)-CiK[x_0(\sec\alpha+1)]\\
&-CiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]-j[2SiK(r_{0l}+l+x_0\sec\alpha)\\
&-SiK[x_0(\sec\alpha+1)]-SiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]]\}\\
&-2e^{jKx_0\cos\alpha}\{CiK(r_{0l}+l+x_0\cos\alpha)-CiK[x_0(1+\cos\alpha)]-j[SiK(r_{0l}+l+x_0\cos\alpha)\\
&-SiK(1+\cos\alpha)]\}+2e^{jK(x_0\cos\alpha-2l\sin^2\alpha)}\{CiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]\\
&-CiK(r_{l0}+x_0\cos\alpha+l\cos2\alpha)-j[SiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]-SiK(r_{l0}+x_0\cos\alpha+l\cos2\alpha)]\}\end{aligned} \qquad (12)$$

Breaking into real and imaginary components,

$$\begin{aligned}-jK\sin^2\alpha\, I_3 &=\\
&\cos K(x_0\sec\alpha)\{2CiK(x_0\sec\alpha+l+\sqrt{x_0^2+l^2+2lx_0\cos\alpha})-CiK[x_0(\sec\alpha+1)]\\
&\qquad -CiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]\}\\
&+\sin K(x_0\sec\alpha)\{2SiK(x_0\sec\alpha+l+\sqrt{x_0^2+l^2+2lx_0\cos\alpha})-SiK[x_0(\sec\alpha+1)]\\
&\qquad -SiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]\}\\
&-\cos K(x_0\cos\alpha)\{CiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l+x_0\cos\alpha)-CiK[x_0(1+\cos\alpha)]\}\\
&-\sin K(x_0\cos\alpha)\{SiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l+x_0\cos\alpha)-SiK[x_0(1+\cos\alpha)]\}\\
&+\cos K(x_0\cos\alpha-2l\sin^2\alpha)\{CiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]-CiK[\sqrt{x_0^2+l^2+2l\cos\alpha x_0}+l\cos2\alpha+x_0\cos\alpha]\}\\
&+\sin K(x_0\cos\alpha-2l\sin^2\alpha)\{SiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]-SiK[\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l\cos2\alpha+x_0\cos\alpha]\}\\
&+j[-\cos K(x_0\sec\alpha)\{2SiK(x_0\sec\alpha+l+\sqrt{x_0^2+l^2+2lx_0\cos\alpha})-SiK[x_0(\sec\alpha+1)]\\
&\qquad -SiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]\}\\
&+\sin K(x_0\sec\alpha)\{2CiK(x_0\sec\alpha+l+\sqrt{x_0^2+l^2+2lx_0\cos\alpha})-CiK[x_0(\sec\alpha+1)]\\
&\qquad -CiK[x_0(\sec\alpha+1)+2l(\cos\alpha+1)]\}\\
&+\cos K(x_0\cos\alpha)\{SiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l+x_0\cos\alpha)-SiK[x_0(1+\cos\alpha)]\}\\
&-\sin K(x_0\cos\alpha)\{CiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l+x_0\cos\alpha)-CiK[x_0(1+\cos\alpha)]\}\\
&-\cos K(x_0\cos\alpha-2l\sin^2\alpha)\{SiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]-SiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l\cos2\alpha+x_0\cos\alpha)\}\\
&+\sin K(x_0\cos\alpha-2l\sin^2\alpha)\{CiK[(x_0+2l\cos\alpha)(1+\cos\alpha)]-CiK(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}+l\cos2\alpha+x_0\cos\alpha)\}\end{aligned} \qquad (13)$$

APPENDIX F

Given $I_4 = \int_0^l \int_0^l e^{jk(x_4+x_1-r_{14})} / r_{14} \, dx_4 dx_1$

$r_{14} = \sqrt{(x_1+x_0\cos\alpha+x_4\cos 2\alpha)^2+(x_0\sin\alpha+x_4\sin 2\alpha)^2}$

$m_1 t_1]_0 = x_0(1-\cos\alpha)$, $m_1 t_1]_l = r_{0l}-x_0\cos\alpha - l\cos 2\alpha$, $r_{00}=x_0$, $r_{0l}=\sqrt{x_0^2+l^2+2lx_0\cos\alpha}$

$m_1 t_2]_0 = r_{l0}-l-x_0\cos\alpha$ $\qquad r_{ll}=x_0+2l\cos\alpha$

$m_1 t_2]_l = r_{ll}-x_0\cos\alpha-2l\cos^2\alpha = (x_0+2l\cos\alpha)(1-\cos\alpha)$

For A_{42}

$m_2 y_1 = x_0(1-\cos\alpha)$, $m_2 y_2 = r_{0l}-l-x_0\cos\alpha$

$u_2 = -k(r_{0l}-l-x_0\sec\alpha)$, $u_1 = -k x_0(1-\sec\alpha)$

For A_{43}

$m_2 y_1 = r_{0l}-x_0\cos\alpha - l\cos 2\alpha$, $m_2 y_2 = (x_0+2l\cos\alpha)(1-\cos\alpha)$

$$u_2 = -k\left(x_0+2l\cos\alpha-x_0\cos\alpha-2l\cos^2\alpha - x_0\frac{\sin^2\alpha}{\cos\alpha}-2l\sin^2\alpha\right)$$
$$= k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]$$

$u_1 = k(x_0\sec\alpha+l-r_{0l})$

$$-jk\sin^2\alpha\, I_4 =$$
$$e^{-jkx_0\sec\alpha}\Big\{2\,\mathrm{Ci}\,k(x_0\sec\alpha+l-r_{0l})-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Ci}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]$$
$$+j\Big[2\,\mathrm{Si}\,k(x_0\sec\alpha+l-r_{0l})-\mathrm{Si}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Si}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\Big]\Big\}$$
$$-e^{-jkx_0\cos\alpha}\Big\{\mathrm{Ci}\,k(r_{0l}-x_0\cos\alpha-l)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]$$
$$-j\big[\mathrm{Si}\,k(r_{0l}-x_0\cos\alpha-l)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\big]\Big\}$$
$$+e^{-jk(x_0\cos\alpha-2l\sin^2\alpha)}\Big\{\mathrm{Ci}\,k[(x_2+2l\cos\alpha)(1-\cos\alpha)]-\mathrm{Ci}\,k(r_{0l}-x_0\cos\alpha-l\cos 2\alpha)$$
$$-j\big[\mathrm{Si}\,k[(x_2+2l\cos\alpha)(1-\cos\alpha)]-\mathrm{Si}\,k[r_{0l}-x_0\cos\alpha-l\cos 2\alpha]\big]\Big\}$$

$$-jk\sin^2\alpha\, I_4 =$$

$$\cos(kx_0\sec\alpha)\Big\{2\,\mathrm{Ci}\,k\big(x_0\sec\alpha+l-\sqrt{x_0^2+l^2+2lx_0\cos\alpha}\big)-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Ci}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\Big\}$$
$$+\sin(kx_0\sec\alpha)\Big\{2\,\mathrm{Si}\,k\big(x_0\sec\alpha+l-\sqrt{x_0^2+l^2+2lx_0\cos\alpha}\big)-\mathrm{Si}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Si}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\Big\}$$
$$-\cos(kx_0\cos\alpha)\Big\{\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\big)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\sin(kx_0\cos\alpha)\Big\{\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\big)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\cos k(x_0\cos\alpha-2l\sin^2\alpha)\Big\{\mathrm{Ci}\,k[(x_0+2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha\big)\Big\}$$
$$-\sin k(x_0\cos\alpha-2l\sin^2\alpha)\Big\{\mathrm{Si}\,k[(x_0+2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha\big)\Big\}$$
$$+j\Big[\cos k(x_0\sec\alpha)\Big\{2\,\mathrm{Si}\,k\big(x_0\sec\alpha+l-\sqrt{x_0^2+l^2+2lx_0\cos\alpha}\big)-\mathrm{Si}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Si}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\Big\}$$
$$-\sin(kx_0\sec\alpha)\Big\{2\,\mathrm{Ci}\,k\big(x_0\sec\alpha+l-\sqrt{x_0^2+l^2+2lx_0\cos\alpha}\big)-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$-\mathrm{Ci}\,k[x_0(\sec\alpha-1)+2l(1-\cos\alpha)]\Big\}$$
$$+\cos(kx_0\cos\alpha)\Big\{\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\big)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$+\sin(kx_0\cos\alpha)\Big\{\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\big)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]\Big\}$$
$$-\cos k(x_0\cos\alpha-2l\sin^2\alpha)\Big\{\mathrm{Si}\,k[(x_0+2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Si}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha\big)\Big\}$$
$$-\sin k(x_0\cos\alpha-2l\sin^2\alpha)\Big\{\mathrm{Ci}\,k[(x_0+2l\cos\alpha)(1-\cos\alpha)]$$
$$-\mathrm{Ci}\,k\big(\sqrt{x_0^2+l^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha\big)\Big\}$$

Given $I_5 = \int_0^{\lambda}\int_0^{l} e^{jK(x_3-x_1-r_{13})}/r_{13}\; dx_3\, dx_1$

$$r_{13} = \sqrt{(x_1+x_0\cos\alpha - x_3\cos 2\alpha)^2 + (x_0\sin\alpha - x_3\sin 2\alpha)^2}$$

$m_1 t = r_{13} + x_1 + x_0\cos\alpha - x_3\cos 2\alpha$, $\quad m_1 = x_0\sin\alpha - x_3\sin 2\alpha$

$m_1 t_1 = r_{03} + x_0\cos\alpha - x_3\cos 2\alpha$, $\quad r_{03} = \sqrt{x_3^2 + x_0^2 - 2x_0x_3\cos\alpha}$

$m_1 t_2 = r_{l3} + l + x_0\cos\alpha - x_3\cos 2\alpha$, $\quad = \sqrt{(x_3 - x_0\cos\alpha)^2 + (x_0\sin\alpha)^2}$

$m_1 t_1]_0 = x_0(1+\cos\alpha)$, $\quad r_{l3} = \sqrt{(x_3 - x_0\cos\alpha - l\cos 2\alpha)^2 + (x_0\sin\alpha + l\sin 2\alpha)^2}$

$m_1 t_1]_l = r_{0l} + x_0\cos\alpha - l\cos 2\alpha$, $\quad r_{00} = x_0,\ r_{0l} = \sqrt{l^2 + x_0^2 - 2lx_0\cos\alpha}$

$m_1 t_2]_0 = r_{l0} + l + x_0\cos\alpha$, $\quad r_{l0} = \sqrt{l^2 + x_0^2 + 2lx_0\cos\alpha}$

$m_1 t_2]_l = r_{ll} + x_0\cos\alpha + 2l\sin^2\alpha$, $\quad r_{ll} = \sqrt{x_0^2 + (2l\sin\alpha)^2}$

$$m_1\, dt = \left[\frac{x_1 + x_0 - x_3\cos\alpha}{r_{13}} + 1\right]dx_1 = \frac{m_1 t}{r_{13}}\,dx_1, \quad dx_1 = \frac{r_{13}}{t}\,dt$$

$$\frac{\partial(m_1 t)}{\partial x_3} = \frac{x_3 - x_0\cos\alpha - x_1\cos 2\alpha}{r_{13}} - \cos 2\alpha = \frac{x_3 - x_0\cos\alpha - (r_{13}+x_1)\cos 2\alpha}{r_{13}}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_3} = \frac{x_3 - x_0\cos\alpha - (r_{l3}+l)\cos 2\alpha}{r_{l3}(r_{l3} + l + x_0\cos\alpha - x_3\cos 2\alpha)}$$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_3} = \frac{x_3 - x_0\cos\alpha - r_{03}\cos 2\alpha}{r_{03}(r_{03} + x_0\cos\alpha - x_3\cos 2\alpha)}$$

$$x_3 - x_1 - r_{13} = -m_1 t + x_0\cos\alpha + 2x_3\sin^2\alpha$$

$$I_5 = e^{jKx_0\cos\alpha}\int_0^l e^{jK2x_3\sin^2\alpha}\int_{Km_1t_1}^{Km_1t_2}\frac{e^{-jt}}{t}\,dt\,dx_3$$

$$= \frac{e^{jKx_0\cos\alpha}}{j2K\sin^2\alpha}\left[e^{j2Kx_3\sin^2\alpha}\left\{\mathrm{Ci}Km_1t_2 - \mathrm{Ci}m_1t_1 - j(\mathrm{Si}Km_1t_2 - \mathrm{Si}Km_1t_1)\right\}\right]_0^l$$

$$-\frac{e^{jKx_0\cos\alpha}}{j2K\sin^2\alpha}\int_0^l e^{jK2x_3\sin^2\alpha}\left\{\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_3} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_3}\right\}dx_3$$

$$A_{51} = -\frac{e^{jKx_0\cos\alpha}}{j2K\sin^2\alpha}\int_0^l e^{jK2x_3\sin^2\alpha}\left\{\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_3} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_3}\right\}dx_3$$

$$Kx_0\cos\alpha + K2x_3\sin^2\alpha - Km_1t_1 = K(x_3 - r_{03})$$

$$Kx_0\cos\alpha + K2x_3\sin^2\alpha - Km_1t_2 = K(x_3 - l - r_{l3})$$

$$A_{51} = -\int_0^l e^{jk(x_3-l-r_{l3})} \frac{[x_3 - x_0\cos\alpha - (r_{l3}+l)\cos 2\alpha]}{r_{l3}[r_{l3}+l+x_0\cos\alpha - x_3\cos 2\alpha]}\, dx_3$$

$$+\int_0^l e^{jk(x_3-r_{03})} \frac{[x_3 - x_0\cos\alpha - r_{03}\cos 2\alpha]}{r_{03}[r_{03}+x_0\cos\alpha - x_3\cos 2\alpha]}\, dx_3$$

$$A_{52} = \int_0^l e^{jk(x_3-r_{03})} \frac{[x_3 - x_0\cos\alpha - r_{03}\cos 2\alpha]}{r_{03}[r_{03}+x_0\cos\alpha - x_3\cos 2\alpha]}\, dx_3$$

$$r_{03} = \sqrt{(x_3 - x_0\cos\alpha)^2 + (x_0\sin\alpha)^2}$$

$$m_2 y = r_{03} - x_3 + x_0\cos\alpha, \quad m_2 = x_0\sin\alpha, \quad x_3 - r_{03} = -m_2 y + x_0\cos\alpha$$

$$m_2 y_2 = r_{0l} - l + x_0\cos\alpha, \quad m_2 y_1 = x_0(1+\cos\alpha) = m_1 t_1 \Big]_0$$

$$m_2\, dy = \left[\frac{x_3 - x_0\cos\alpha}{r_{03}} - 1\right] dx_3 = -\frac{m_2 y}{r_{03}}\, dx_3, \quad dx_3 = -\frac{r_{03}\, dy}{y}$$

$$\frac{m_2}{y} = \frac{(x_0\sin\alpha)^2}{r_{03} - x_3 + x_0\cos\alpha} = r_{03} + x_3 - x_0\cos\alpha, \quad x_0\cos\alpha(1-\cos 2\alpha) = m_2\sin 2\alpha$$

$$r_{03} = \frac{m_2}{2y}(y^2+1), \quad x_3 - x_0\cos\alpha = -\frac{m_2}{2y}(y^2-1)$$

$$A_{52} = e^{jkx_0\cos\alpha}\int_{y_1}^{y_2} \frac{e^{-jkm_2 y}\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right] dy}{y\left[\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + m_2\sin 2\alpha\right]}$$

$$= e^{jkx_0\cos\alpha}\int_{y_1}^{y_2}\left(\frac{2}{y+\tan\alpha} - \frac{1}{y}\right) e^{-jkm_2 y}\, dy$$

$$u = km_2(y+\tan\alpha) = km_2 y + kx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$-jkm_2 y = -ju + jkx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$u_2 = k(r_{0l} - l + x_0\sec\alpha), \quad u_1 = kx_0(\sec\alpha + 1)$$

$$u = km_2 y, \quad u_2 = k(r_{0l} - l + x_0\cos\alpha), \quad u_1 = kx_0(1+\cos\alpha)$$

$$A_{52} = 2e^{jkx_0\sec\alpha}\Big\{Cik(r_{0l} - l + x_0\sec\alpha) - Cik[x_0(\sec\alpha+1)]$$
$$-j\big[Sik(r_{0l} - l + x_0\sec\alpha) - Sik[x_0(\sec\alpha+1)]\big]\Big\}$$
$$-e^{jkx_0\cos\alpha}\Big\{Cik(r_{0l} - l + x_0\cos\alpha) - Cik[x_0(1+\cos\alpha)]$$
$$-j\big[Sik(r_{0l} - l + x_0\cos\alpha) - Sik[x_0(1+\cos\alpha)]\big]\Big\}$$

$$A_{53} = -\int_0^l e^{jk(x_3 - l - r_{l3})} \frac{[x_3 - x_0\cos\alpha - (r_{l3}+l)\cos 2\alpha]}{r_{l3}[r_{l3} + l + x_0\cos\alpha - x_3\cos 2\alpha]} dx_3$$

$$r_{l3} = \sqrt{(x_3 - x_0\cos\alpha - l\cos 2\alpha)^2 + (x_0\sin\alpha + l\sin 2\alpha)^2}$$

$$m_2 y = r_{l3} - x_3 + x_0\cos\alpha + l\cos 2\alpha, \quad m_2 = x_0\sin\alpha + l\sin 2\alpha$$

$$x_3 - l - r_{l3} = -m_2 y + x_0\cos\alpha - 2l\sin^2\alpha, \; r_{l0} = \sqrt{l^2 + x_0^2 + 2lx_0\cos 2\alpha}$$

$$m_2 y_2 = r_{ll} + x_0\cos\alpha - 2l\sin^2\alpha \quad , \; r_{ll} = \sqrt{x_0^2 + (2l\sin\alpha)^2}$$

$$m_2 y_1 = r_{l0} + x_0\cos\alpha + l\cos 2\alpha$$

$$m_2 dy = \left[\frac{x_3 - x_0\cos\alpha - l\cos 2\alpha}{r_{l3}} - 1\right]dx_3 = -\frac{m_2 y}{r_{l3}}dx_3, \; dx_3 = -\frac{r_{l3}\,dy}{y}$$

$$\frac{m_2}{y} = \frac{(x_0\sin\alpha + l\sin 2\alpha)^2}{r_{l3} - x_3 + x_0\cos\alpha + l\cos 2\alpha} = r_{l3} + x_3 - x_0\cos\alpha - l\cos 2\alpha$$

$$r_{l3} = \frac{m_2}{2y}(y^2+1) \qquad l + x_0\cos\alpha(1-\cos 2\alpha) - l^2\cos 2\alpha =$$

$$x_3 - x_0\cos\alpha - l\cos 2\alpha = -\frac{m_2}{2y}(y^2-1) \qquad 2x_0\cos\alpha\sin^2\alpha + l\sin^2 2\alpha = m_2\sin 2\alpha$$

$$A_{53} = -e^{jk(x_0\cos\alpha - 2l\sin^2\alpha)} \int_{y_1}^{y_2} \frac{e^{-jkm_2 y}\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right]dy}{y\left[\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha + m_2\sin 2\alpha\right]}$$

$$= -e^{jk(x_0\cos\alpha - 2l\sin^2\alpha)} \int_{y_1}^{y_2} e^{-jkm_2 y}\left[\frac{2}{y+\tan\alpha} - \frac{1}{y}\right]dy$$

$$u = km_2(y + \tan\alpha) = k\left(m_2 y + x_0\frac{\sin^2\alpha}{\cos\alpha} + 2l\sin^2\alpha\right), \; -jkm_2 y = -ju + jk\left(x_0\frac{\sin^2\alpha}{\cos\alpha} + 2l\sin^2\alpha\right)$$

$$u_2 = r_{ll} + x_0\sec\alpha \quad , \; u_1 = r_{l0} + x_0\sec\alpha + l$$

$$u = km_2 y, \; u_2 = k(r_{ll} + x_0\cos\alpha - 2l\sin^2\alpha), \; u_1 = r_{l0} + x_0\cos\alpha + l\cos 2\alpha$$

$$A_{53} = -2e^{jkx_0\sec\alpha}\Big\{\mathrm{Ci}k(r_{ll} + x_0\sec\alpha) - \mathrm{Ci}k(r_{l0} + x_0\sec\alpha + l)$$
$$- j\left[\mathrm{Si}k(r_{ll} + x_0\sec\alpha) - \mathrm{Si}k(r_{l0} + x_0\sec\alpha + l)\right]\Big\}$$
$$+ e^{jk(x_0\cos\alpha - 2l\sin^2\alpha)}\Big\{\mathrm{Ci}k(r_{ll} + x_0\cos\alpha - 2l\sin^2\alpha) - \mathrm{Ci}k(r_{l0} + x_0\cos\alpha + l\cos 2\alpha)$$
$$- j\left[\mathrm{Si}k(r_{ll} + x_0\cos\alpha - 2l\sin^2\alpha) - \mathrm{Si}k(r_{l0} + x_0\cos\alpha + l\cos 2\alpha)\right]\Big\}$$

$$j2k\sin^2\alpha\, I_5 = 2e^{jkx_0\sec\alpha}\{Cik(r_{0\ell}-\ell+x_0\sec\alpha)+Cik(r_{\ell 0}+\ell+x_0\sec\alpha)$$
$$-Cik[x_0(\sec\alpha+1)]-Cik(r_{\ell\ell}+x_0\sec\alpha)-j[Sik(r_{0\ell}-\ell+x_0\sec\alpha)+Sik(r_{\ell 0}+\ell+x_0\sec\alpha)$$
$$-Sik[x_0(\sec\alpha+1)]-Sik(r_{\ell\ell}+x_0\sec\alpha)]\}-e^{jkx_0\cos\alpha}\{Cik(r_{0\ell}-\ell+x_0\cos\alpha)$$
$$+Cik(r_{\ell 0}+\ell+x_0\cos\alpha)-2Cik[x_0(1+\cos\alpha)]-j[Sik(r_{0\ell}-\ell+x_0\cos\alpha)$$
$$+Sik(r_{\ell 0}+\ell+x_0\cos\alpha)-2Sik[x_0(1+\cos\alpha)]]\}+e^{jk(x_0\cos\alpha+2\ell\sin^2\alpha)}$$
$$\{Cik(r_{\ell\ell}+x_0\cos\alpha+2\ell\sin^2\alpha)-Cik(r_{0\ell}+x_0\cos\alpha-\ell\cos2\alpha)-j[Sik(r_{\ell\ell}+x_0\cos\alpha+2\ell\sin^2\alpha)$$
$$-Sik(r_{0\ell}+x_0\cos\alpha-\ell\cos2\alpha)]\}+e^{jk(x_0\cos\alpha-2\ell\sin^2\alpha)}\{Cik(r_{\ell\ell}+x_0\cos\alpha-2\ell\sin^2\alpha)$$
$$-Cik(r_{\ell 0}+x_0\cos\alpha+\ell\cos2\alpha)-j[Sik(r_{\ell\ell}+x_0\cos\alpha-2\ell\sin^2\alpha)-Sik(r_{\ell 0}+x_0\cos\alpha+\ell\cos2\alpha)]\}$$

$$j2k\sin^2\alpha\, I_5 = 2\cos(kx_0\sec\alpha)\{Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\sec\alpha)$$
$$+Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\sec\alpha)-Cik[x_0(\sec\alpha+1)]-Cik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\sec\alpha)\}$$
$$+2\sin(kx_0\sec\alpha)\{Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\sec\alpha)+Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\sec\alpha)$$
$$-Sik[x_0(\sec\alpha+1)]-Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\sec\alpha)\}$$
$$-\cos(kx_0\cos\alpha)\{Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\cos\alpha)+Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\cos\alpha)-2Cik[x_0(1+\cos\alpha)]\}$$
$$-\sin(kx_0\cos\alpha)\{Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\cos\alpha)+Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\cos\alpha)-2Sik[x_0(1+\cos\alpha)]\}$$
$$+\cos k(x_0\cos\alpha+2\ell\sin^2\alpha)\{Cik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha+2\ell\sin^2\alpha)-Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}+x_0\cos\alpha-\ell\cos2\alpha)\}$$
$$+\sin k(x_0\cos\alpha+2\ell\sin^2\alpha)\{Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha+2\ell\sin^2\alpha)-Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}+x_0\cos\alpha-\ell\cos2\alpha)\}$$
$$+\cos k(x_0\cos\alpha-2\ell\sin^2\alpha)\{Cik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha-2\ell\sin^2\alpha)-Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+x_0\cos\alpha+\ell\cos2\alpha)\}$$
$$+\sin k(x_0\cos\alpha-2\ell\sin^2\alpha)\{Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha-2\ell\sin^2\alpha)-Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+x_0\cos\alpha+\ell\cos2\alpha)\}$$
$$+j[-2\cos(kx_0\sec\alpha)\{Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\sec\alpha)+Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\sec\alpha)$$
$$-Sik[x_0(\sec\alpha+1)]-Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\sec\alpha)\}$$
$$+2\sin(kx_0\sec\alpha)\{Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\sec\alpha)+Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\sec\alpha)$$
$$+\cos(kx_0\cos\alpha)\{Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\cos\alpha)+Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\cos\alpha)$$
$$-2Sik[x_0(1+\cos\alpha)]\}-\sin(kx_0\cos\alpha)\{Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}-\ell+x_0\cos\alpha)$$
$$+Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+\ell+x_0\cos\alpha)-2Cik[x_0(1+\cos\alpha)]\}$$
$$-\cos k(x_0\cos\alpha+2\ell\sin^2\alpha)\{Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha+2\ell\sin^2\alpha)-Sik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}+x_0\cos\alpha-\ell\cos2\alpha)\}$$
$$+\sin k(x_0\cos\alpha+2\ell\sin^2\alpha)\{Cik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha+2\ell\sin^2\alpha)-Cik(\sqrt{\ell^2+x_0^2-2\ell x_0\cos\alpha}+x_0\cos\alpha-\ell\cos2\alpha)\}$$
$$-\cos k(x_0\cos\alpha-2\ell\sin^2\alpha)\{Sik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha-2\ell\sin^2\alpha)-Sik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+x_0\cos\alpha+\ell\cos2\alpha)\}$$
$$+\sin k(x_0\cos\alpha-2\ell\sin^2\alpha)\{Cik(\sqrt{x_0^2+(2\ell\sin\alpha)^2}+x_0\cos\alpha-2\ell\sin^2\alpha)-Cik(\sqrt{\ell^2+x_0^2+2\ell x_0\cos\alpha}+x_0\cos\alpha+\ell\cos2\alpha)\}$$

$$I_7 = \int_0^l \int_0^l \frac{e^{-jK(x_2-x_1-r_{12})}}{r_{12}}\, dx_2\, dx_1 , \quad r_{12} = \sqrt{(x_1-x_0-x_2\cos 2\alpha)^2+(x_0\sin\alpha+x_2\sin 2\alpha)^2}$$

$$m_1 t = r_{12}+x_1-x_0\cos\alpha - x_2\cos 2\alpha , \quad m_1 = x_0\sin\alpha + x_2 \sin 2\alpha$$

$$x_2-x_1-r_{12} = -m_1 t - x_0\cos\alpha + 2x_2\sin^2\alpha , \quad r_{02} = \sqrt{x_2^2+x_0^2+2x_0x_2\cos\alpha}$$

$$= \sqrt{(x_2+x_0\cos\alpha)^2+(x_0\sin\alpha)^2}$$

$$m_1 t_2 = r_{l2}+l-x_0\cos\alpha - x_2\cos 2\alpha$$

$$m_1 t_1 = r_{02}-x_0\cos\alpha - x_2\cos 2\alpha$$

$$r_{l2} = \sqrt{(x_2+x_0\cos\alpha - l\cos 2\alpha)^2+(x_0\sin\alpha - l\sin 2\alpha)^2}$$

$$m_1 t_1\big]_l = r_{0l} - x_0\cos\alpha - l\cos 2\alpha$$

$$r_{00} = x_0$$

$$m_1 t_2\big]_l = r_{ll} - x_0\cos\alpha + 2l\sin^2\alpha$$

$$r_{0l} = \sqrt{l^2+x_0^2+2lx_0\cos\alpha}$$

$$r_{l0} = \sqrt{l^2+x_0^2-2lx_0\cos\alpha}$$

$$m_1 t_1\big]_0 = x_0(1-\cos\alpha)$$

$$r_{ll} = \sqrt{x_0^2+(2l\sin\alpha)^2}$$

$$m_1 t_2\big]_0 = r_{l0} - x_0\cos\alpha + l$$

$$m_1\, dt = \left[\frac{x_1-x_0\cos\alpha-x_2\cos 2\alpha}{r_{12}}+1\right] = \frac{m_1 t}{r_{12}}\, dx_1 , \quad dx_1 = \frac{r_{12}\, dt}{t}$$

$$\frac{\partial(m_1 t)}{\partial x_2} = \frac{x_2+x_0\cos\alpha - x_1\cos 2\alpha}{r_{12}} - \cos 2\alpha = \frac{x_2+x_0\cos\alpha-(r_{12}+x_1\cos 2\alpha)}{r_{12}}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_2} = \frac{x_2+x_0\cos\alpha-(r_{l2}+l)\cos 2\alpha}{r_{l2}(r_{l2}+l-x_0\cos\alpha-x_2\cos 2\alpha)}$$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_2} = \frac{x_2+x_0\cos\alpha - r_{02}\cos 2\alpha}{r_{02}(r_{02}-x_0\cos\alpha-x_2\cos 2\alpha)}$$

$$I_7 = e^{-jKx_0\cos\alpha}\int_0^l e^{j2Kx_2\sin^2\alpha}\int_{Km_1t_1}^{Km_1t_2}\frac{e^{-jt}}{t}\, dt$$

$$= \frac{e^{-jKx_0\cos\alpha}}{j2K\sin^2\alpha}\left[e^{j2Kx_2\sin^2\alpha}\left\{\mathrm{Ci}\,Km_1t_2 - \mathrm{Ci}\,Km_1t_1 - j(\mathrm{Si}\,Km_1t_2 - \mathrm{Si}\,Km_1t_1)\right\}\right]_0^l$$

$$- \frac{e^{-jKx_0\cos\alpha}}{j2K\sin^2\alpha}\int_0^l e^{j2Kx_2\sin^2\alpha}\left\{\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2}\right\} dx_2$$

$$A_{71} = -\frac{e^{-jKx_0\cos\alpha}}{j2K\sin^2\alpha}\int_0^l e^{j2Kx_2\sin^2\alpha}\left\{\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2}\right\} dx_2$$

$$-Kx_0\cos\alpha + 2Kx_2\sin^2\alpha - Km_1t_2 = K(x_2-l-r_{l2})$$

$$-Kx_0\cos\alpha + 2Kx_2\sin^2\alpha - Km_1t_1 = K(x_2-r_{02})$$

$$A_{71} = -\int_0^l \frac{e^{jK(x_2-l-r_{l2})}\left[x_2+x_0\cos\alpha-(r_{l2}+l)\cos 2\alpha\right]}{r_{l2}\left[r_{l2}+l-x_0\cos\alpha-x_2\cos 2\alpha\right]}\, dx_2$$

$$+\int_0^l \frac{e^{jK(x_2-r_{02})}\left[x_2+x_0\cos\alpha-r_{02}\cos 2\alpha\right]}{r_{02}\left[r_{02}-x_0\cos\alpha-x_2\cos 2\alpha\right]}\, dx_2$$

$$A_{72}=\int_0^l \frac{e^{jk(x_2-r_{02})}(x_2+x_0\cos\alpha-r_{02}\cos 2\alpha)}{r_{02}(r_{02}-x_0\cos\alpha-x_2\cos 2\alpha)}dx_2,\quad r_{02}=\sqrt{(x_2+x_0\cos\alpha)^2+(x_0\sin\alpha)^2}$$

$$m_2 y=r_{02}-x_2-x_0\cos\alpha,\quad x_2-r_{02}=-m_2y-x_0\cos\alpha,\quad m_2=x_0\sin\alpha$$

$$m_2y_2=r_{0l}-x_0\cos\alpha-l,\quad m_2y_1=x_0(1-\cos\alpha)=m_1t_1]_0$$

$$m_2\,dy=\left[\frac{x_2+x_0\cos\alpha}{r_{02}}-1\right]dx_2=-\frac{m_2y}{r_{02}}dx_2,\quad dx_2=-\frac{r_{02}}{y}dy$$

$$m_2/y=\frac{(x_0\sin\alpha)^2}{r_{02}-x_0\cos\alpha-x_2}=r_{02}+x_0\cos\alpha+x_2,\quad r_{02}=\frac{m_2}{2y}(y^2+1)$$

$$x_2+x_0\cos\alpha=-\frac{m_2}{2y}(y^2-1),\quad -x_0(1-\cos 2\alpha)\cos\alpha=-2x_0\sin^2\alpha\cos\alpha=-m_2\sin 2\alpha$$

$$A_{72}=e^{-jkx_0\cos\alpha}\int_{y_1}^{y_2}\frac{e^{-jkm_2y}}{y}\frac{\left[\frac{m_2}{2y}(y^2-1)+\frac{m_2}{2y}(y^2+1)\cos 2\alpha\right]dy}{\left[\frac{m_2}{2y}(y^2+1)+\frac{m_2}{2y}(y^2-1)\cos 2\alpha-m_2\sin 2\alpha\right]}$$

$$=e^{-jkx_0\cos\alpha}\int_{y_1}^{y_2}\left(\frac{2}{y-\tan\alpha}-\frac{1}{y}\right)e^{-jkm_2y}dy$$

$$-u=km_2(y-\tan\alpha)=km_2y-kx_0\frac{\sin^2\alpha}{\cos\alpha},\quad -jkm_2y=ju-jkx_0\frac{\sin^2\alpha}{\cos\alpha}$$

$$-u_2=k(r_{0l}-x_0\sec\alpha-l),\quad +u_1=-kx_0(1-\sec\alpha)$$

$$u=km_2y,\quad u_2=r_{0l}-x_0\cos\alpha-l,\quad u_1=x_0(1-\cos\alpha)$$

$$A_{72}=2e^{-jkx_0\sec\alpha}\Big\{\mathrm{Ci}\,k(x_0\sec\alpha+l-r_{0l})-\mathrm{Ci}\,k[x_0(\sec\alpha-1)]$$
$$+j\left[\mathrm{Si}\,k(x_0\sec\alpha+l-r_{0l})-\mathrm{Si}\,k[x_0(\sec\alpha-1)]\right]\Big\}$$
$$-e^{-jkx_0\cos\alpha}\Big\{\mathrm{Ci}\,k(r_{0l}-x_0\cos\alpha-l)-\mathrm{Ci}\,k[x_0(1-\cos\alpha)]$$
$$-j\left[\mathrm{Si}\,k(r_{0l}-x_0\cos\alpha-l)-\mathrm{Si}\,k[x_0(1-\cos\alpha)]\right]\Big\}$$

$$A_{73}=-\int_0^l\frac{e^{jk(x_2-l-r_{l2})}[x_2+x_0\cos\alpha-(r_{l2}+l)\cos 2\alpha]\,dx_2}{r_{l2}\,[r_{l2}+l-x_0\cos\alpha-x_2\cos 2\alpha]}$$

$$r_{l2}=\sqrt{(x_2+x_0\cos\alpha-l\cos 2\alpha)^2+(x_0\sin\alpha-l\sin 2\alpha)^2}$$

$$m_2y=r_{l2}-x_2-x_0\cos\alpha+l\cos 2\alpha,\quad m_2=x_0\sin\alpha-l\sin 2\alpha$$

$$x_2-l-r_{l2}=-x_0\cos\alpha-2l\sin^2\alpha-m_2y,\quad r_{l0}=\sqrt{l^2+x_0^2-2lx_0\cos\alpha}$$

$$m_2y_2=r_{ll}-x_0\cos\alpha-2l\sin^2\alpha,\quad r_{ll}=\sqrt{x_0^2+(2l\sin\alpha)^2}$$

$$m_2y_1=r_{l0}-x_0\cos\alpha+l\cos 2\alpha$$

$$\frac{m_2}{y} = \frac{(x_0\sin\alpha - l\sin 2\alpha)^2}{r_{l2} - x_2 - x_0\cos\alpha + l\cos 2\alpha} = r_{l2} + x_2 + x_0\cos\alpha - l\cos 2\alpha$$

$$r_{l2} = \frac{m_2}{2y}(y^2+1)\ ,\ (x_0\cos\alpha - l\cos 2\alpha)\cos 2\alpha + l - x_0\cos\alpha =$$
$$l\sin^2 2\alpha - 2x_0\cos\alpha\sin^2\alpha = -m_2\sin 2\alpha$$

$$x_2 + x_0\cos\alpha - l\cos 2\alpha = -\frac{m_2}{2y}(y^2-1)$$

$$A_{\eta_3} = -e^{-jk(x_0\cos\alpha + 2l\sin^2\alpha)}\int_{y_1}^{y_2}\frac{e^{-jkm_2y}}{y}\frac{\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right]dy}{\left[\frac{m_2}{2y}(y^2+1) + \frac{m_2}{2y}(y^2-1)\cos 2\alpha - m_2\sin 2\alpha\right]}$$

$$= -e^{-jk(x_0\cos\alpha + 2l\sin^2\alpha)}\int_{y_1}^{y_2}\left(\frac{2}{y-\tan\alpha} - \frac{1}{y}\right)e^{-jkm_2y}\,dy$$

$$u = -km_2(y - \tan\alpha) = -k(m_2y - x_0\frac{\sin^2\alpha}{\cos\alpha} + 2l\sin^2\alpha)$$

$$-km_2y = u - k(x_0\frac{\sin^2\alpha}{\cos\alpha} - 2l\sin^2\alpha)$$

$$u_2 = k(x_0\sec\alpha - r_{ll})\ ,\ u_1 = k(x_0\sec\alpha - r_{l0} - l)$$

$$u = -km_2y\ ,\ u_2 = r_{ll} - x_0\cos\alpha - 2l\sin^2\alpha,\ u_1 = r_{l0} - x_0\cos\alpha + l\cos 2\alpha$$

$$A_{\eta_3} = -2e^{-jkx_0\sec\alpha}\{Ci\,k(x_0\sec\alpha - r_{ll}) - Ci\,k(x_0\sec\alpha - l - r_{l0})$$
$$+ j[Si\,k(x_0\sec\alpha - r_{ll}) - Si\,k(x_0\sec\alpha - l - r_{l0})]\}$$
$$+ e^{-jk(x_0\cos\alpha + 2l\sin^2\alpha)}\{Ci\,k(r_{ll} - x_0\cos\alpha - 2l\sin^2\alpha) - Ci\,k(r_{l0} - x_0\cos\alpha + l\cos 2\alpha)$$
$$-j[Si\,k(r_{ll} - x_0\cos\alpha - 2l\sin^2\alpha) - Si\,k(r_{l0} - x_0\cos\alpha + l\cos 2\alpha)]\}$$

$$j2k\sin^2\alpha\, I_\eta = 2e^{-jkx_0\sec\alpha}\{Ci\,k(x_0\sec\alpha + l - r_{0l}) + Ci\,k(x_0\sec\alpha - l - r_{l0})$$
$$-Ci\,k[x_0(\sec\alpha - 1)] - Ci\,k(x_0\sec\alpha - r_{ll}) + j[Si\,k(x_0\sec\alpha + l - r_{0l})$$
$$+ Si\,k(x_0\sec\alpha - l - r_{l0}) - Si\,k[x_0(\sec\alpha - 1)] - Si\,k(x_0\sec\alpha - r_{ll})]\}$$
$$-e^{-jkx_0\cos\alpha}\{Ci\,k(r_{0l} - l - x_0\cos\alpha) + Ci\,k(r_{l0} + l - x_0\cos\alpha) - 2Ci\,k[x_0(1-\cos\alpha)]$$
$$-j[Si\,k(r_{0l} - l - x_0\cos\alpha) + Si\,k(r_{l0} + l - x_0\cos\alpha) - 2Si\,k[x_0(1-\cos\alpha)]]\}$$
$$+ e^{-jk(x_0\cos\alpha + 2l\sin^2\alpha)}\{Ci\,k(r_{ll} - x_0\cos\alpha - 2l\sin^2\alpha) - Ci\,k(r_{l0} - x_0\cos\alpha + l\cos 2\alpha)$$
$$-j[Si\,k(r_{ll} - x_0\cos\alpha - 2l\sin^2\alpha) - Si\,k(r_{l0} - x_0\cos\alpha + l\cos 2\alpha)]\}$$
$$+ e^{-jk(x_0\cos\alpha - 2l\sin^2\alpha)}\{Ci\,k(r_{ll} - x_0\cos\alpha + 2l\sin^2\alpha) - Ci\,k(r_{0l} - x_0\cos\alpha - l\cos 2\alpha)$$
$$-j[Si\,k(r_{ll} - x_0\cos\alpha + 2l\sin^2\alpha) - Si\,k(r_{0l} - x_0\cos\alpha - l\cos 2\alpha)]\}$$

$j2K\sin^2 I_7 =$

$2\cos(Kx_0\sec\alpha)\{\mathrm{Ci}K(x_0\sec\alpha + l - \sqrt{l^2+x_0^2+2lx_0\cos\alpha})$

$+\mathrm{Ci}K(x_0\sec\alpha - l - \sqrt{l^2+x_0^2+2lx_0\cos\alpha}) - \mathrm{Ci}K[x_0(\sec\alpha-1)] - \mathrm{Ci}K(x_0\sec\alpha - \sqrt{x_0^2+(2l\sin\alpha)^2})\}$

$+2\sin(Kx_0\sec\alpha)\{\mathrm{Si}K(x_0\sec\alpha+l-\sqrt{l^2+x_0^2+2lx_0\cos\alpha}) - \mathrm{Si}K[x_0(\sec\alpha-1)]$

$+\mathrm{Si}K(x_0\sec\alpha - l - \sqrt{l^2+x_0^2-2lx_0\cos\alpha}) - \mathrm{Si}K(x_0\sec\alpha - \sqrt{x_0^2+(2l\sin\alpha)^2})\}$

$-\cos(Kx_0\cos\alpha)\{\mathrm{Ci}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-l-x_0\cos\alpha)+\mathrm{Ci}K(\sqrt{l^2+x_0^2-2lx_0\cos\alpha}+l-x_0\cos\alpha)$

$-2\mathrm{Ci}K[x_0(1-\cos\alpha)]\}$

$+\sin(Kx_0\cos\alpha)\{\mathrm{Si}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-l-x_0\cos\alpha)+\mathrm{Si}K(\sqrt{l^2+x_0^2-2lx_0\cos\alpha}+l-x_0\cos\alpha)$

$-2\mathrm{Si}K[x_0(1-\cos\alpha)]\}$

$+\cos K(x_0\cos\alpha+2l\sin^2\alpha)\{\mathrm{Ci}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha-2l\sin^2\alpha)-\mathrm{Ci}K(\sqrt{x_0^2+l^2-2lx_0\cos\alpha}-x_0\cos\alpha+l\cos2\alpha)\}$

$-\sin K(x_0\cos\alpha+2l\sin^2\alpha)\{\mathrm{Si}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha-2l\sin^2\alpha)-\mathrm{Si}K(\sqrt{l^2+x_0^2-2lx_0\cos\alpha}-x_0\cos\alpha+l\cos2\alpha)\}$

$+\cos K(x_0\cos\alpha-2l\sin^2\alpha)\{\mathrm{Ci}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha+2l\sin^2\alpha)-\mathrm{Ci}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha)\}$

$-\sin K(x_0\cos\alpha-2l\sin^2\alpha)\{\mathrm{Si}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha+2l\sin^2\alpha)-\mathrm{Si}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha)\}$

$+j\Big[2\cos(Kx_0\sec\alpha)\{\mathrm{Si}K(x_0\sec\alpha+l-\sqrt{l^2+x_0^2+2lx_0\cos\alpha})-\mathrm{Si}K[x_0(\sec\alpha-1)]$

$+\mathrm{Si}K(x_0\sec\alpha-l-\sqrt{l^2+x_0^2-2lx_0\cos\alpha})-\mathrm{Si}K(x_0\sec\alpha-\sqrt{x_0^2+(2l\sin\alpha)^2})\}$

$-2\sin(Kx_0\sec\alpha)\{\mathrm{Ci}K(x_0\sec\alpha+l-\sqrt{l^2+x_0^2+2lx_0\cos\alpha})-\mathrm{Ci}K[x_0(\sec\alpha-1)]$

$+\mathrm{Ci}K(x_0\sec\alpha-l-\sqrt{l^2+x_0^2-2lx_0\cos\alpha})-\mathrm{Ci}K(x_0\sec\alpha-\sqrt{x_0^2+(2l\sin\alpha)^2})\}$

$+\cos(Kx_0\cos\alpha)\{\mathrm{Si}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-l-x_0\cos\alpha)-2\mathrm{Si}K[x_0(1-\cos\alpha)]$

$+\mathrm{Si}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}+l-x_0\cos\alpha)\}$

$-\sin(Kx_0\cos\alpha)\{\mathrm{Ci}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-l-x_0\cos\alpha)-2\mathrm{Ci}K[x_0(1-\cos\alpha)]$

$+\mathrm{Ci}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}+l-x_0\cos\alpha)\}$

$-\cos K(x_0\cos\alpha+2l\sin^2\alpha)\{\mathrm{Si}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha-2l\sin^2\alpha)-\mathrm{Si}K(\sqrt{l^2+x_0^2-2lx_0\cos\alpha}-x_0\cos\alpha+l\cos2\alpha)\}$

$-\sin K(x_0\cos\alpha+2l\sin^2\alpha)\{\mathrm{Ci}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha-2l\sin^2\alpha)-\mathrm{Ci}K(\sqrt{l^2+x_0^2-2lx_0\cos\alpha}-x_0\cos\alpha+l\cos2\alpha)\}$

$-\cos K(x_0\cos\alpha-2l\sin^2\alpha)\{\mathrm{Si}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha+2l\sin^2\alpha)-\mathrm{Si}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha)\}$

$-\sin K(x_0\cos\alpha-2l\sin^2\alpha)\{\mathrm{Ci}K(\sqrt{x_0^2+(2l\sin\alpha)^2}-x_0\cos\alpha+2l\sin^2\alpha)-\mathrm{Ci}K(\sqrt{l^2+x_0^2+2lx_0\cos\alpha}-x_0\cos\alpha-l\cos2\alpha)\}\Big]$

APPENDIX I

For $x_0 = 2l\cos\alpha + b$, $\quad b \ll l$

$$2l + b\sec\alpha - l - \sqrt{4l^2\cos^2\alpha + 4lb\cos\alpha + b^2 - 4l^2\cos^2\alpha - 2lb\cos\alpha + l^2} =$$

$$l + b\sec\alpha - \sqrt{b^2 + 2lb\cos\alpha + l^2} = l + b\sec\alpha - l\left[1 + \frac{b^2 + 2lb\cos\alpha}{l^2}\right]^{1/2} =$$

$$l + b\sec\alpha - l\left[1 + \frac{b^2 + 2lb\cos\alpha}{2l^2} - \frac{b^4 + 4lb^3\cos\alpha + 4l^2b^2\cos^2\alpha}{8l^4}\right] =$$

$$l + b\sec\alpha - \left[l + b\cos\alpha + \frac{b^2\sin^2\alpha}{2l}\right] = b(\sec\alpha - \cos\alpha) - \frac{b^2\sin^2\alpha}{2l} =$$

$$b\frac{\sin^2\alpha}{\cos\alpha}$$

$$2l + b\sec\alpha - \sqrt{4l^2\cos^2\alpha + 4lb\cos\alpha + 4l^2\sin^2\alpha} =$$

$$2l + b\sec\alpha - \sqrt{4l^2 + 4lb\cos\alpha + b^2} =$$

$$2l + b\sec\alpha - 2l\left[1 + \frac{b^2 + 4lb\cos\alpha}{4l^2}\right]^{1/2} =$$

$$2l + b\sec\alpha - 2l\left[1 + \frac{b^2 + 4lb\cos\alpha}{8l^2} - \frac{b^4 + 8lb^3 + 16l^2b^2\cos^2\alpha}{128l^4}\right] =$$

$$2l + b\sec\alpha - 2l - \frac{4lb\cos\alpha + b^2\sin^2\alpha}{4l} = b(\sec\alpha - \cos\alpha)$$

$$= \frac{b\sin^2\alpha}{\cos\alpha}$$

$$\sqrt{4l^2\cos^2\alpha + 4lb\cos\alpha + b^2 + 4l\sin^2\alpha} - 2l\cos^2\alpha - b\cos\alpha - 2l\sin^2\alpha =$$

$$2l + \frac{4lb\cos\alpha + b^2\sin^2\alpha}{4l} - 2l - b\cos\alpha = \frac{b^2\sin^2\alpha}{4l}$$

$$l + b\cos\alpha + \frac{b^2\sin^2\alpha}{2l} - 2l\cos^2\alpha - b\cos\alpha + 2l\cos^2\alpha - l = \frac{b^2\sin^2\alpha}{2l}$$